Mathematical Foundations of computer science

Lecture 5: Completeness in propositional logic

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Recap

Our first goal: formalize a "mathematician"

- Analysis target of Mathematician $\approx$ Formulas
  - Mathematical assertions $\to$ Formulas
  - Truth of assertions $\to$ Valuation function

- Mathematician $\approx$ those who write proofs of formulas
  - Tools: axioms, assumptions, inference rules
  - Proof $\to$ seq. of formulas from axioms/assump. to target, connected by inference rules

- Properties of Mathematician
  - Soundness theorem: proved formulas are "true"
  - Today: *Which "true" formulas can Mathematician prove?*

Today's main theorem: completeness

Theorem (Soundness of proof structure)

If there is a proof of φ from Σ, then Σ logically implies φ; that is,

$$\Sigma \vdash \varphi \textit{ implies } \Sigma \models \varphi.$$

The theorem roughly claims "correctness" of Mathematician:
a proved formula under assumptions Σ is true whenever Σ is.

Theorem ((First) completeness of proof structure)

$$\Sigma \models \varphi \textit{ implies } \Sigma \vdash \varphi.$$

The theorem roughly claims the "capability" of Mathematician:
They can prove *any* φ from Σ whenever Σ logically implies φ.

Today's main theorem: completeness

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Consistency and inconsistency

Definition

We say that Σ is **inconsistent** if $\Sigma \vdash \bot$. If $\Sigma \nvdash \bot$ then we say that Σ is **consistent**.

A proof of a formula φ from inconsistent Σ implies nothing nontrivial about the truth of φ (next page).

Thus, consistency could be said as a minimum requirement to Σ as reasonable assumptions by Mathematician.

Note: (In)consistency is a purely syntactic notion, i.e., it does not involve valuation of Σ in its definition.

What does inconsistency mean?

Lemma

There is no model of inconsistent Σ.

Assume that Σ is inconsistent.

Then $\Sigma \vdash \bot$.

By the soundness theorem, any model of Σ should be a model of $\bot$.

But, $\bot$ has no model.
Hence, all inconsistent sets Σ have no models.

What does inconsistency mean?

Lemma

Inconsistent Σ *can prove any formula, that is,* $\Sigma \vdash \varphi$ *for any* φ.

Proving this lemma is left as a homework.

Second Completeness theorem

"Σ has a model" could be seen as a minimum requirement to Σ as assumptions by Mathematician, *via the semantic notion*.

- If not, then $\Sigma \models \varphi$ trivially holds for any φ
- For the *syntactic* notion, consistency of Σ does the job

The following theorem says these two requirements coincide.

Theorem (The Second Completeness Theorem)

A set of formulas Σ is consistent iff Σ has a model.

The MP rule over provability

Let Σ be a set of formulas, and φ and ψ be formulas.

Lemma (The MP rule over provability)

If $\Sigma \vdash \varphi$ *and* $\Sigma \vdash \varphi \to \psi$*, then* $\Sigma \vdash \psi$*.*

Proof: Make a sequence of formulas by appending the proof of φ, then the proof of $\varphi \to \psi$, and then the formula ψ. This is a proof of ψ.

The Deduction Lemma

Let Σ be a set of formulas, and φ and ψ be formulas.

Lemma (Deduction Lemma)

If $\Sigma \cup \{\varphi\} \vdash \psi$ *then* $\Sigma \vdash \varphi \to \psi$*.*

Proof. Induction of the length of the proof $\Sigma \cup \{\varphi\} \vdash \psi$.

Assume the length is 1. Then ψ is an axiom or $\psi \in \Sigma$ or $\varphi = \psi$. In the first two cases, $\Sigma \vdash \psi$ and $\psi \to (\varphi \to \psi)$ is an axiom. By the MP we have $\Sigma \vdash \varphi \to \psi$. If $\varphi = \psi$, then we already have $\vdash \varphi \to \varphi$ and hence $\Sigma \vdash \varphi \to \psi$.

The Deduction Lemma: Inductive Step

Let $\varphi_1, \ldots, \varphi_{k+1}$ be a proof of ψ from $\Sigma \cup \{\varphi\}$. So, $\varphi_{k+1} = \psi$. If φ_{k+1} is an axiom or $\varphi_{k+1} \in \Sigma$ or $\varphi_{k+1} \in \{\varphi_1, \ldots, \varphi_k\}$, then the inductive hypothesis is applied, and we are done.

Otherwise, we have $\Sigma \cup \{\varphi\} \vdash \alpha$ and $\Sigma \cup \{\varphi\} \vdash \alpha \to \psi$, where $(\alpha \to \psi), \alpha \in \{\varphi_1, \ldots, \varphi_k\}$.

By the inductive hypothesis, $\Sigma \vdash \varphi \to (\alpha \to \psi)$ and $\Sigma \vdash \varphi \to \alpha$.

We have the axiom $(\varphi \to (\alpha \to \psi)) \to ((\varphi \to \alpha) \to (\varphi \to \psi))$. Applying the MP rule twice we get $\Sigma \vdash \varphi \to \psi$.

The Deduction Lemma: Application 1

Corollary (proof by contradiction)

$\Sigma \vdash \varphi$ *if and only if* $\Sigma \cup \{\neg\varphi\}$ *is inconsistent.*

Proof.

Assume $\Sigma \vdash \varphi$. Consider $\varphi \to (\neg\varphi \to \bot)$ an axiom. By the MP, we have $\Sigma \vdash \neg\varphi \to \bot$. Also, $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$. Again, by the MP, we have $\Sigma \cup \{\neg\varphi\} \vdash \bot$.

Assume $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Then $\Sigma \cup \{\neg\varphi\} \vdash \bot$. By the deduction lemma, $\Sigma \vdash \neg\varphi \to \bot$. We have $(\neg\varphi \to \bot) \to \varphi$ axiom. By the MP rule, we get $\Sigma \vdash \varphi$. □

The Deduction Lemma: Application 2

Corollary

If $\Sigma \vdash \varphi$ and Σ is consistent then $\Sigma \cup \{\varphi\}$ is also consistent.

Proof.

Assume the contrary, i.e., assume $\Sigma \cup \{\varphi\}$ is inconsistent. Then $\Sigma \cup \{\varphi\} \vdash \bot$. By the deduction lemma, we have $\Sigma \vdash \varphi \to \bot$. By the MP rule, $\Sigma \vdash \bot$. This contradicts consistency of Σ; hence, $\Sigma \cup \{\varphi\}$ is consistent. □

Complete sets

Definition

A set of formulas Σ is **complete** if Σ is consistent and for any formula φ either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.

Intuitively, a complete Σ is "the book of truth"; you can take any φ and ask Σ, "is φ true or not?". Then Σ says yes if $\varphi \in \Sigma$; or no if $\neg\varphi \in \Sigma$. (the precise meaning of "truth" will be formally characterized shortly.)

Lemma (Lindenbaum Lemma)

Any set Σ of consistent formulas has a complete extension.

Proof of Lindenbaum Lemma

List all formulas $\varphi_1, \varphi_2, \ldots, \varphi_i, \ldots$.

For each φ_i, we need to put either φ_i or $\neg\varphi_i$ into Σ. Set $\Sigma_0 = \Sigma$.

We build the sequence $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_i \subseteq \ldots$ as follows. By assumption Σ_0 is consistent. Consider φ_{n+1}:

- If $\Sigma_n \vdash \varphi_{n+1}$ then set $\Sigma_{n+1} = \Sigma_n \cup \{\varphi_{n+1}\}$.
- If $\Sigma_n \not\vdash \varphi_{n+1}$ then set $\Sigma_{n+1} = \Sigma_n \cup \{\neg\varphi_{n+1}\}$.

By the two corollaries, Σ_{n+1} is consistent. Let $\Sigma_\infty = \bigcup_{n \in \mathbb{N}} \Sigma_n$ (i.e., the union of Σ_n for all n). Then Σ_∞ is consistent (why?).

By construction, for each φ_i, $\varphi_i \in \Sigma_\infty$ or $\neg\varphi_i \in \Sigma_\infty$. Therefore, Σ_∞ is a complete extension of Σ.

Building a model for complete set Σ

Let Σ be a complete set. Since Σ is complete, for each $p \in P$, either $p \in \Sigma$ or $\neg p \in \Sigma$. Define $A_\Sigma : P \to \{\textbf{true}, \textbf{false}\}$:

$$A_\Sigma(p) = \textbf{true} \text{ if } p \in \Sigma, \text{ and otherwise } A_\Sigma(p) = \textbf{false}.$$

Let V_Σ be the valuation under A_Σ (i.e., $V_\Sigma = V_{A_\Sigma}$).

Now our intuition in the previous slide is formalized below: Σ *is "the book of truth under* A_Σ*"*.

Lemma

The mapping V_Σ *has the following property. For all* $\varphi \in FORM$*:*

$$\varphi \in \Sigma \ \textit{ if and only if } \ V_\Sigma(\varphi) = \textbf{true}.$$

In particular A_Σ *is a model of* Σ*.*

Proof of the lemma (by induction on structure of φ)

If $\varphi = p \in P$, then $V_\Sigma(\varphi) = A_\Sigma(\varphi) =$ **true** iff $\varphi \in \Sigma$.

Case 1: $\varphi = \neg\psi$. We have the following:

$$\begin{aligned} \varphi \in \Sigma &\Leftrightarrow \psi \notin \Sigma && \text{(completeness of } \Sigma\text{)} \\ &\Leftrightarrow V_\Sigma(\psi) = \textbf{false} && \text{(induction hypothesis)} \\ &\Leftrightarrow V_\Sigma(\varphi) = \textbf{true} && \text{(def. of valuation)} \end{aligned}$$

Proof of the lemma (by induction on structure of φ)

Case 2: $\varphi = \psi_1 \vee \psi_2$. We have the following:

$$\begin{aligned} \varphi \in \Sigma &\Leftrightarrow \psi_1 \in \Sigma \text{ or } \psi_2 \in \Sigma && \text{(homework)} \\ &\Leftrightarrow V_\Sigma(\psi_1) = \textbf{true} \text{ or } V_\Sigma(\psi_2) = \textbf{true} && \text{(induction hyp.)} \\ &\Leftrightarrow V_\Sigma(\varphi) = \textbf{true} && \text{(def. of valuation)} \end{aligned}$$

Case 3: $\varphi = \psi_1 \,\&\, \psi_2$. Left to the reader.

The Second Completeness Theorem

Theorem (The Second Completeness Theorem)

A set of formulas Σ is consistent iff Σ has a model.

Proof.

Let Σ be consistent. Then it has a complete extension Σ'. By the lemma above, Σ' has a model. Hence Σ too has a model.

Assume that Σ has a model. Assume that Σ is inconsistent. Then $\Sigma \vdash \bot$. Then the model must make $\bot$ **true** (why?). This is a contradiction. □

The First Completeness Theorem

Theorem (The First Completeness Theorem)

If $\Sigma \models \varphi$ *then* $\Sigma \vdash \varphi$*.*

Proof.

Since $\Sigma \models \varphi$, we have $\Sigma \cup \{\neg\varphi\}$ has no model. So, $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Hence, $\Sigma \vdash \varphi$. □

Compactness Theorem

Theorem (Compactness Theorem)

If $\Sigma \models \varphi$ *then there is a finite subset* Σ_0 *of* Σ *such that* $\Sigma_0 \models \varphi$*.*

Proof.

The First Completeness Theorem implies $\Sigma \vdash \varphi$. So, there is a finite subset Σ_0 of Σ such that $\Sigma_0 \vdash \varphi$. This implies $\Sigma_0 \models \varphi$. $\square$

Corollary

Σ *has a model iff every finite subset of* Σ *has a model.*

Proof.

The direction $\rightarrow$ is clear. Every finite subset of Σ is consistent as it has a model. So, Σ is consistent. Thus, Σ has a model. $\square$

Conclusion

1. $\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$ (Soundness and Completeness). So $\vdash$ and $\models$ are equivalent concepts.
2. If $\Sigma \vdash \varphi$ then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \varphi$ (Proofs are algorithmic).
3. If Σ has a model iff all finite subsets $\Sigma_0 \subset \Sigma$ have models (Compactness).