Ranking and Repulsing Supermartingales for Reachability in Probabilistic Programs

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ERATO 蓮尾メタ数理システムデザインプロジェクト

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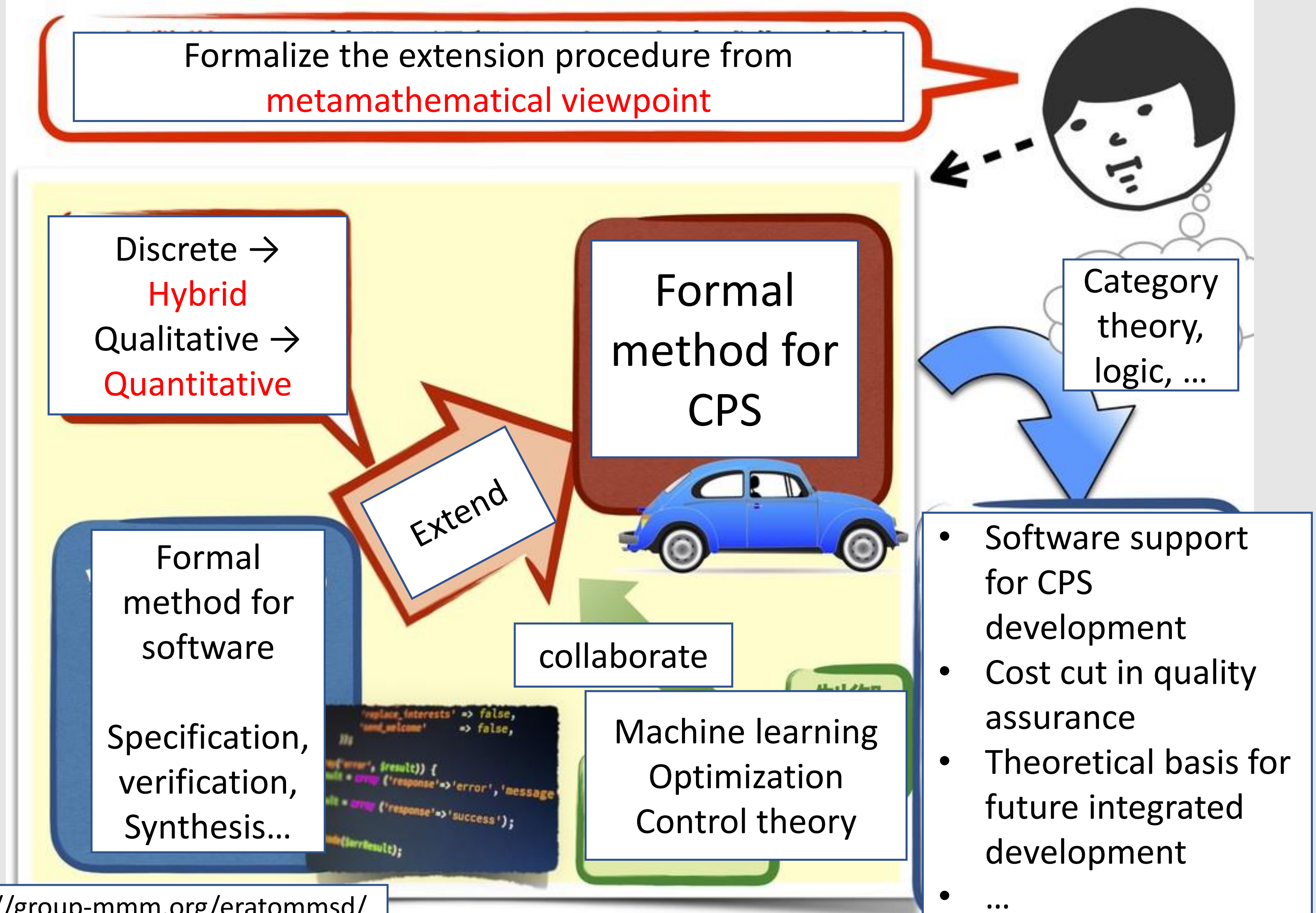

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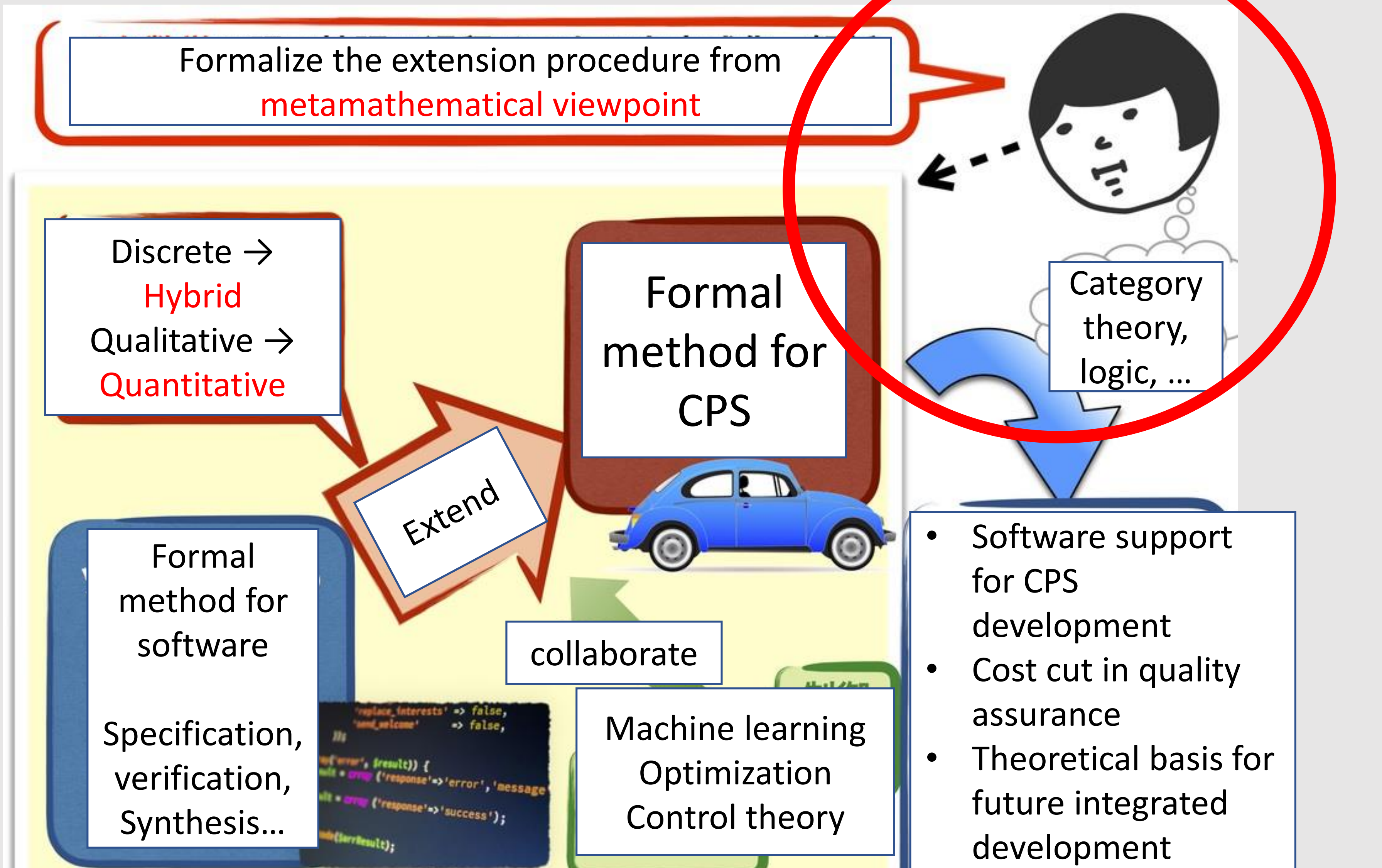

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Outline

- Introduction / preliminaries
  - Our topic: supermartingale for reachability analysis
  - What can supermartingale do?
  - What is supermartingale? / Why does it work?
  - Which property of SM techniques are we interested? - Soundness / completeness

- Our contribution
  - Theoretical part: characterization of SM techniques via KT theorem
  - Implementation and experiments

Problem formulation

Input: probabilistic program

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Nondet. / Prob. branching

Nondet. / Prob. assignment

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What is the probability that the program terminates?

(under angelic/demonic scheduler)

We admit continuous variables

⇒ Generally one can't compute probability efficiently

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⇒ Reachability analysis by supermartingale

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Ranking supermartingale for a.s. termination (Chakarov-Sankaranarayanan, CAV'13 etc.)

Probabilistic modification of real-world benchmarks
(in Alias+, SAS'10)

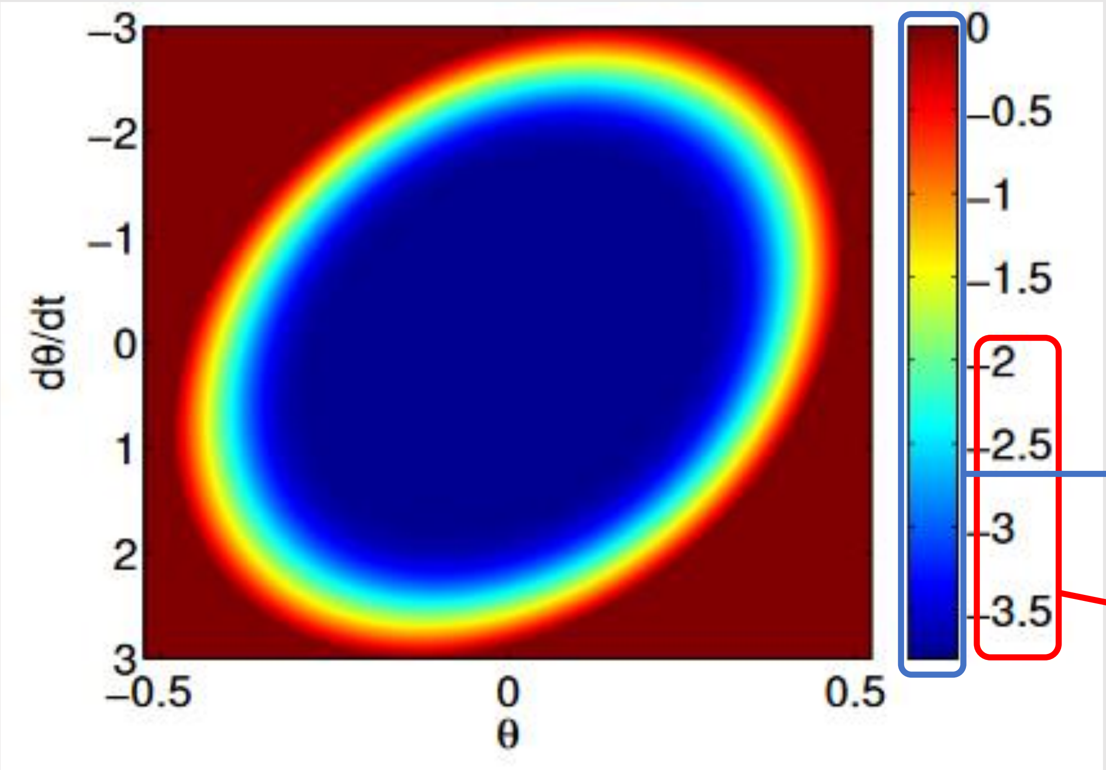
A.s. termination is certified in 20/28 examples

Benchmark	Time (s)	Solution	Dimension	Prob. loops	Prob. Assignments
alain	0.11	yes	2	yes	yes
catmouse	0.08	yes	2	yes	yes
counterex1a	0.1	no		no	no
counterex1c	0.11	yes	3	yes	yes
easy1	0.09	yes	1	yes	yes
exmini	0.09	yes	2	yes	yes
insertsort	0.1	yes	3	yes	yes
ndecr	0.09	yes	2	yes	yes
perfect	0.11	yes	3	yes	yes
perfect2	0.1	yes	3	yes	no
	0.11	no		yes	yes
real2	0.09	no		no	no
realbubble	0.22	yes	3	yes	yes
realselect	0.11	yes	3	yes	yes
realshellsort	0.09	no		yes	no
serpent	0.1	yes	1	yes	yes
sipmabubble	0.1	yes	3	yes	yes
speedDis2	0.09	no		no	no
speedNestedMultiple	0.1	yes	3	yes	yes
speedpldi2	0.09	yes	2	yes	yes
speedpldi4	0.09	yes	3	yes	yes
speedSimpleMultipleDep	0.09	no		no	no
speedSingleSingle2	0.12	yes	2	yes	no
	0.1	no		yes	yes
unperfect	0.1	yes	2	yes	no
	0.16	no		yes	yes
wcet1	0.11	yes	2	yes	yes
while2	0.1	yes	3	yes	yes

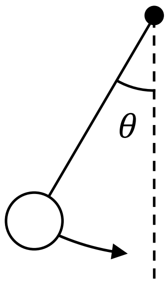
(Agrawal+, POPL'18)

Repulsing supermartingale for lower bound of safety probability

(Steinhardt-Tedrake, IJRR'12; Chatterjee+, POPL'17 etc.)



System: pendulum + noise



Failure ⇔

$\theta > \pi/6$ at time $t \leq$ 1hour

The log-base-10 of the failure probability

>99% safety is guaranteed

(Pr(failure) <1%)

(Steinhardt-Tedrake, IJRR'12)

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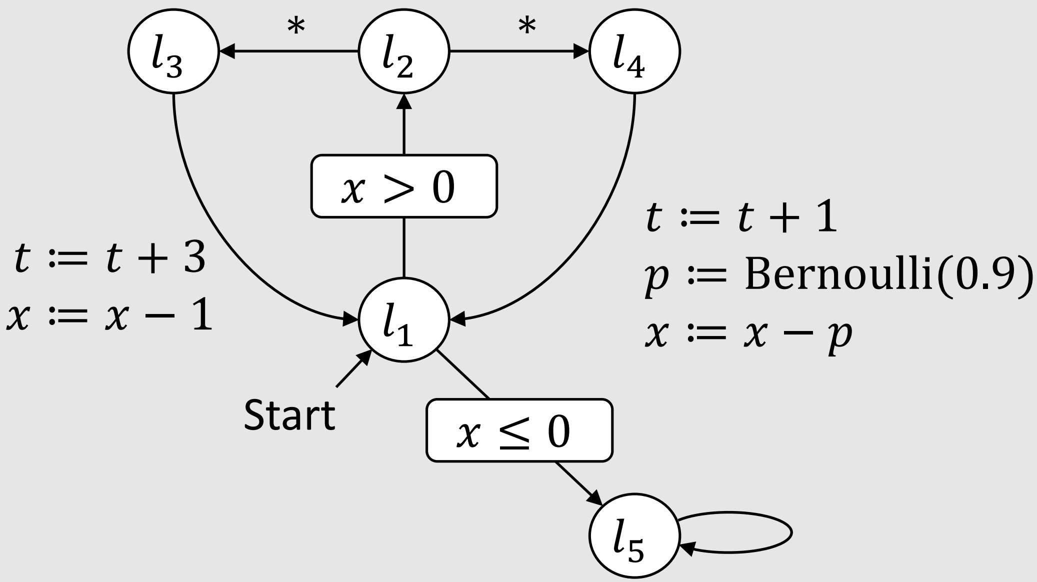
Semantics: Control flow graph

(Agrawal+, POPL'18 etc.)

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- A state is a pair (program location, memory state)
- Nondet. / prob. branching

finite

$\mathbb{R}^V$

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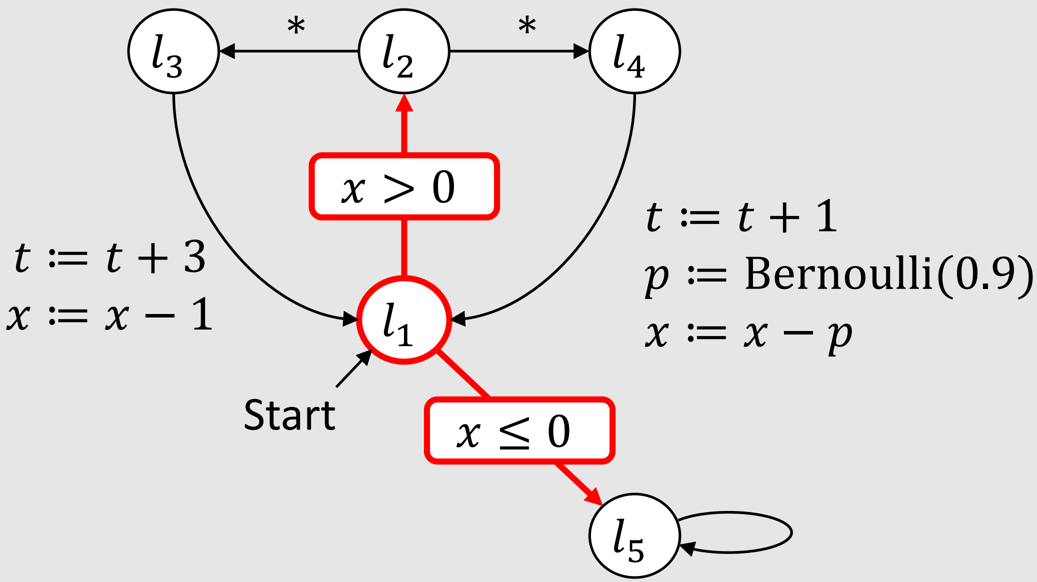


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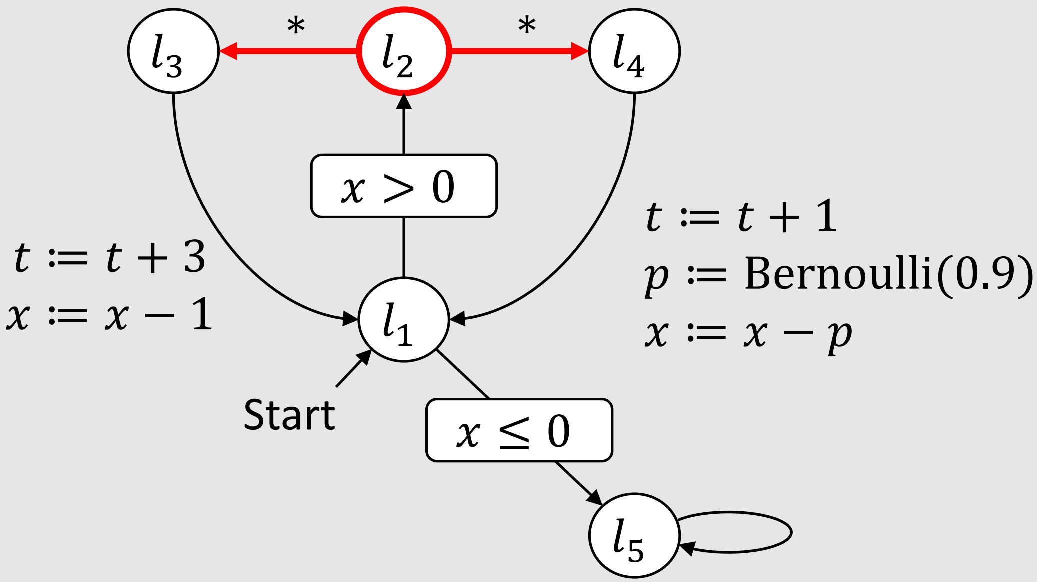


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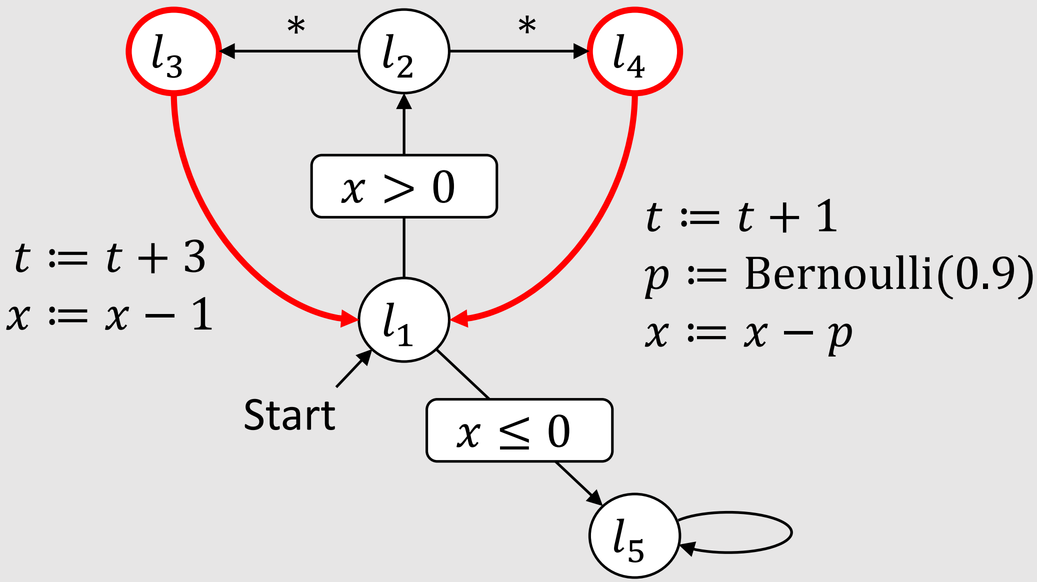


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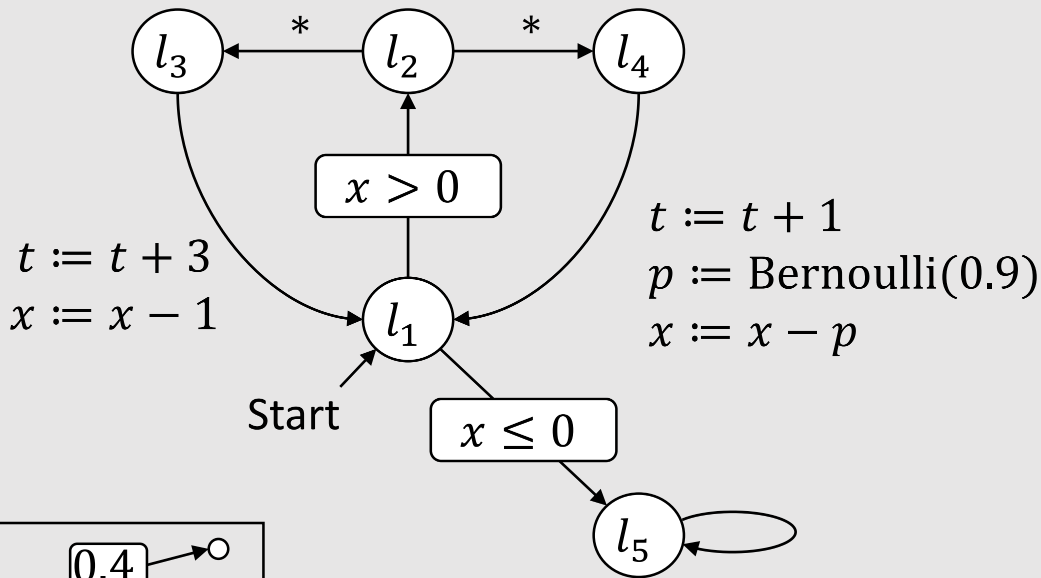


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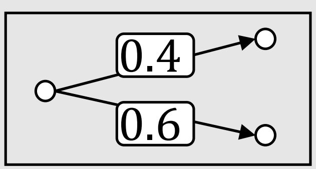

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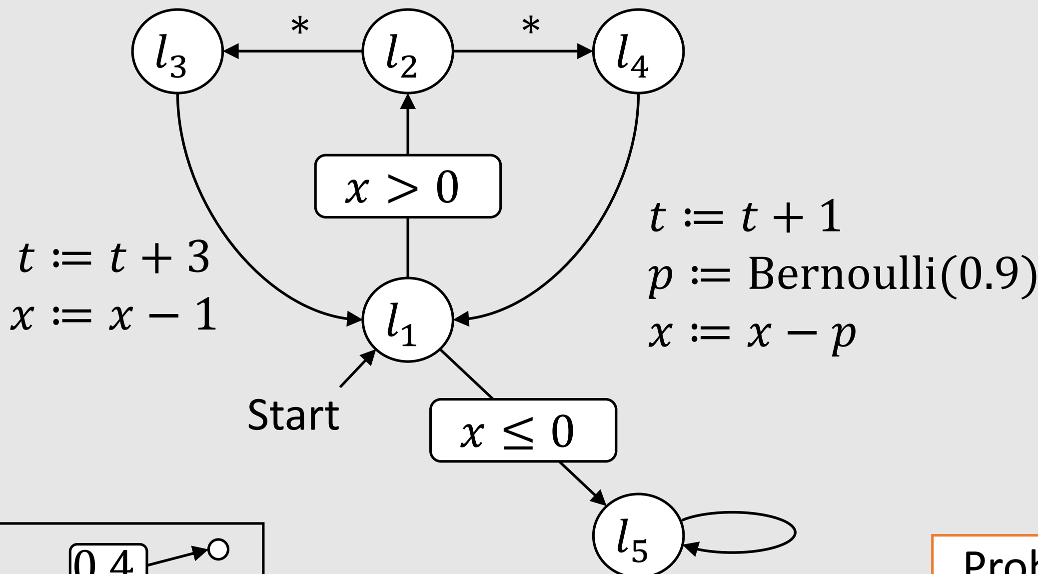


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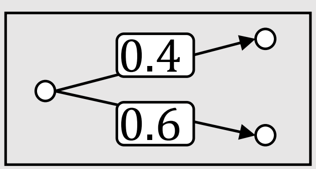

Problem

C = (terminating states)

$= \{l_5\} \times \{(x, t, p) \mid t \leq 20\}$

⇒Pr(the system eventually visits the region C)?

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$\mathbb{R}^V$

Semantics: Control flow graph

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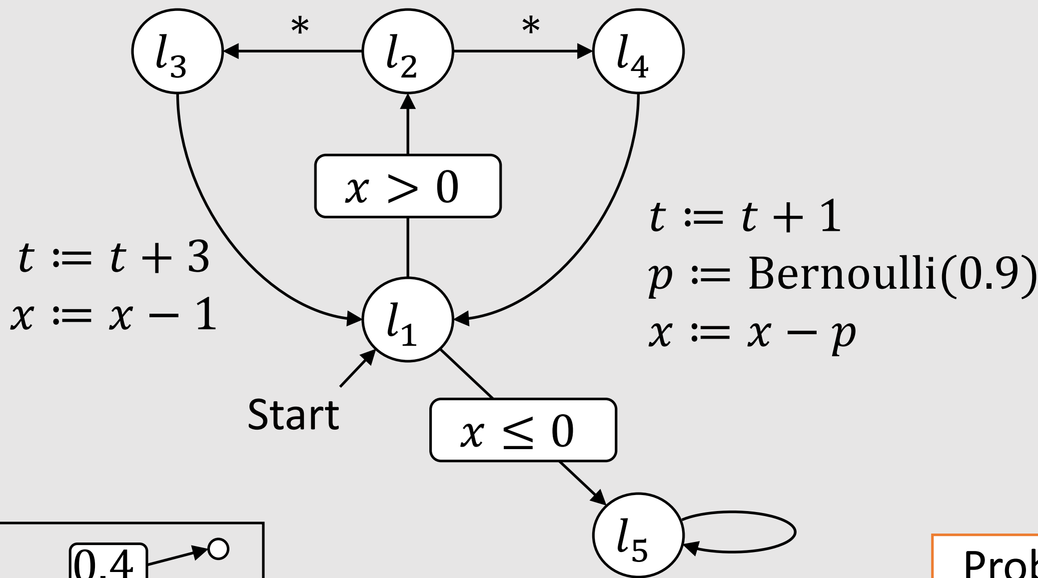


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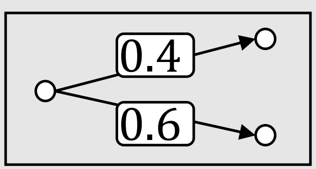

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...under angelic/demonic scheduler

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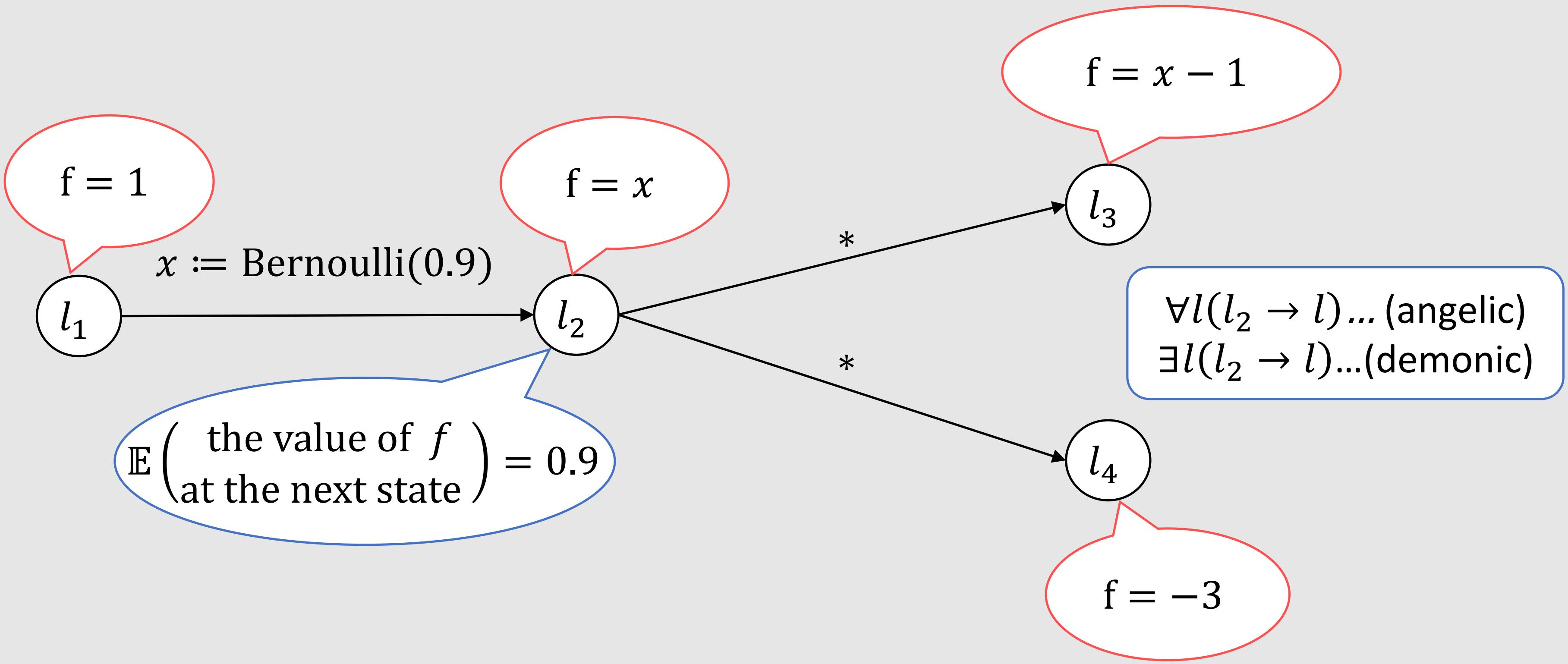
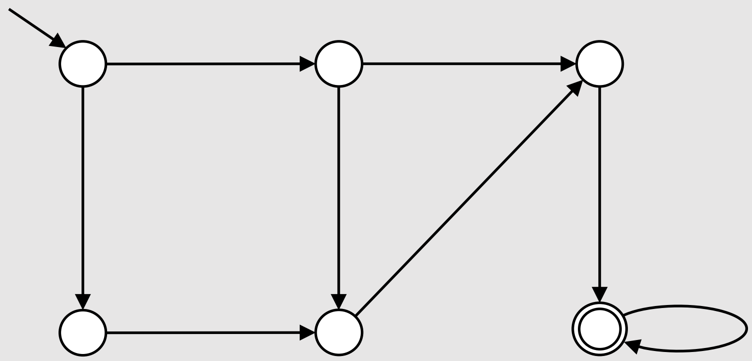
Supermartingale = a function over states that is "non-increasing" through transitions



Ranking function



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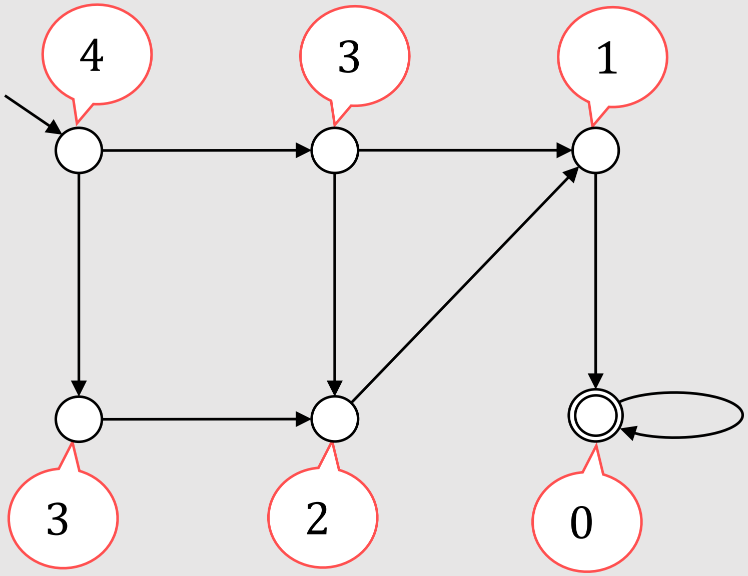

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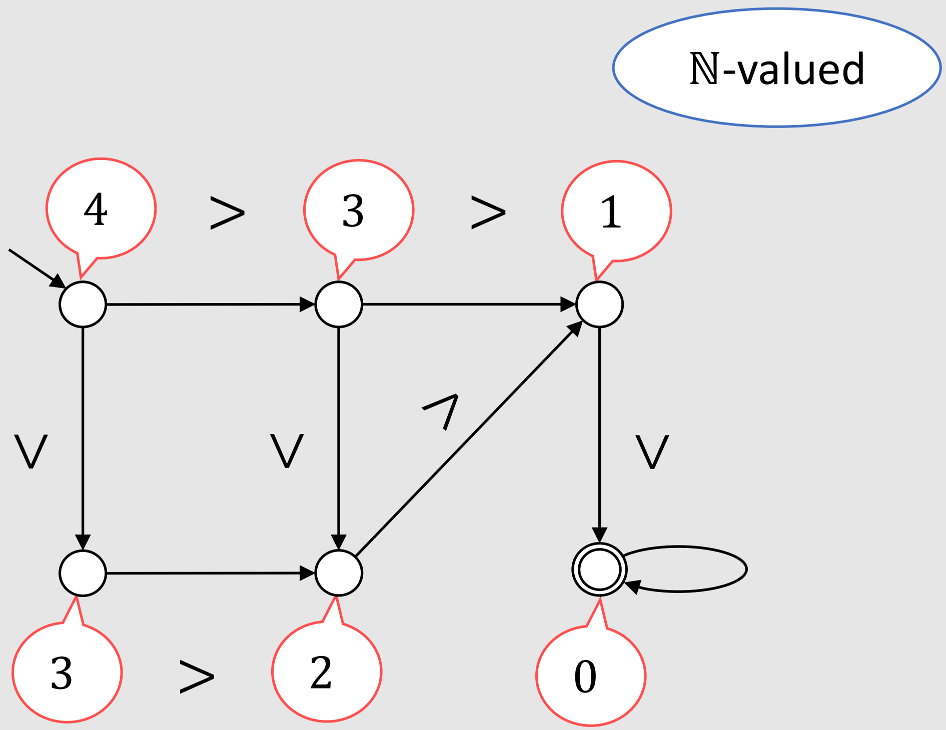

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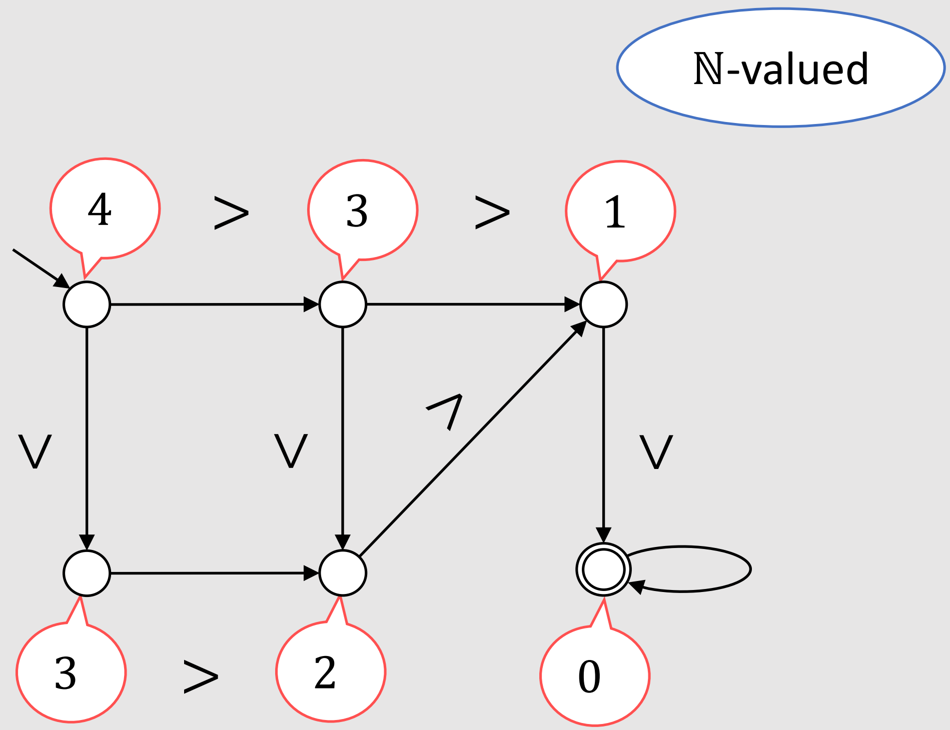
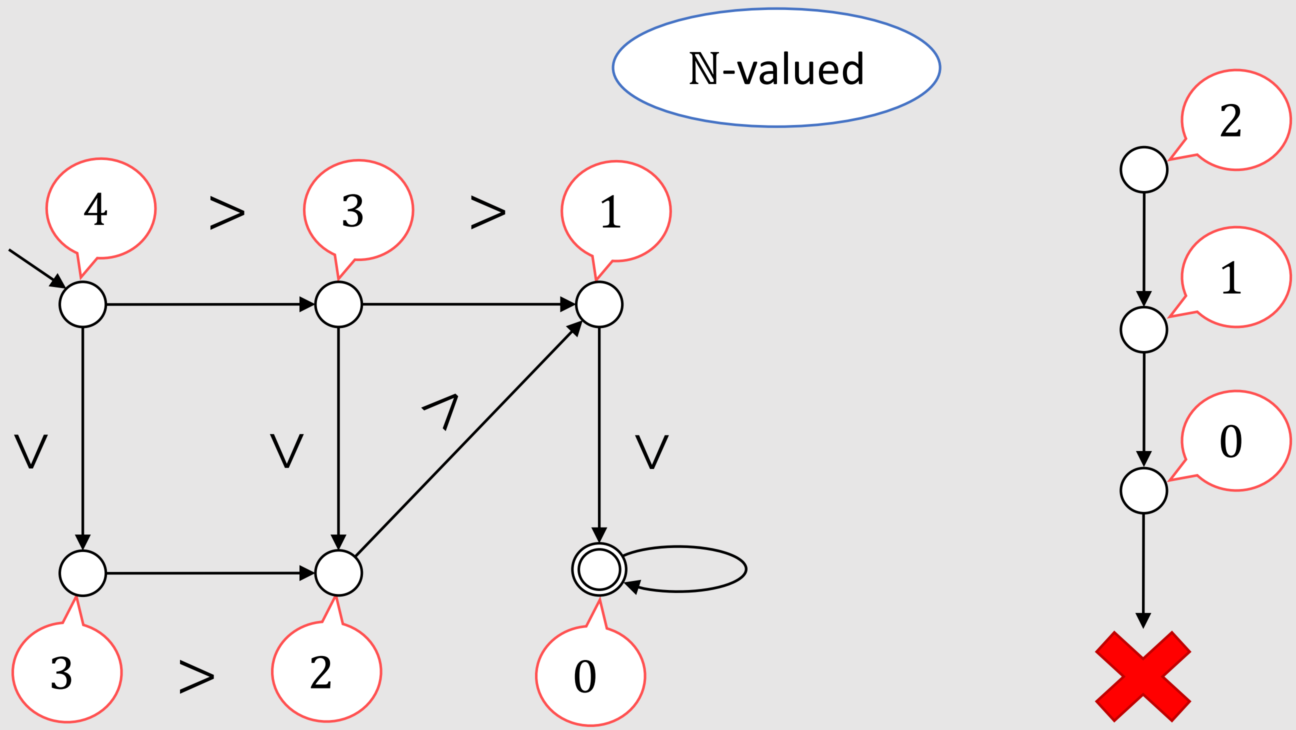


The system eventually visits ◎ (under any nondeterministic choice)

Ranking function



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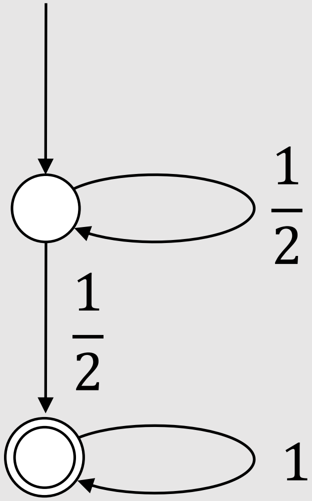
Ranking supermartingale



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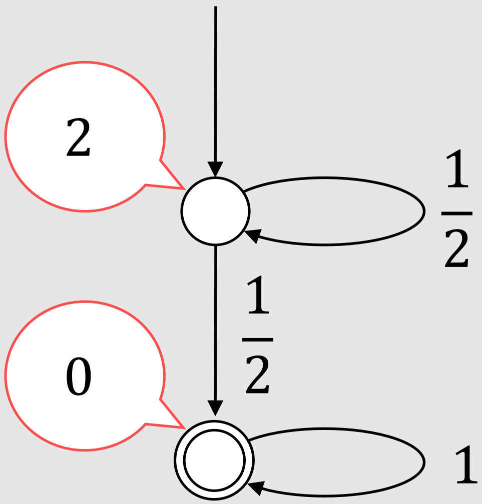

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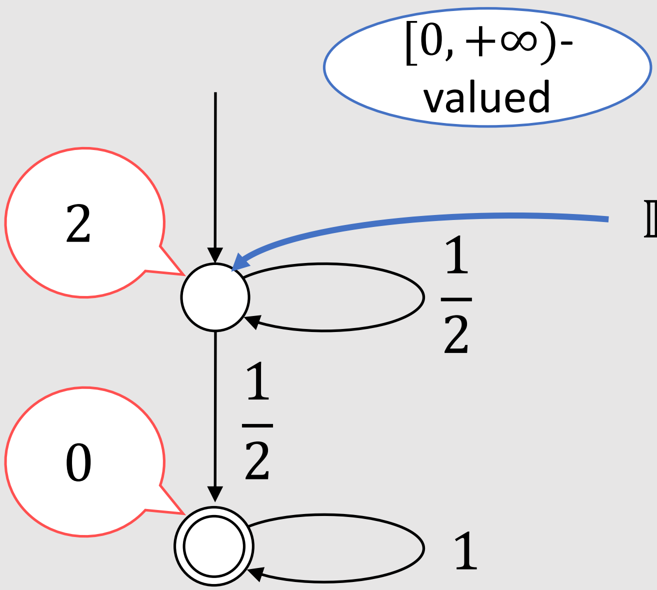
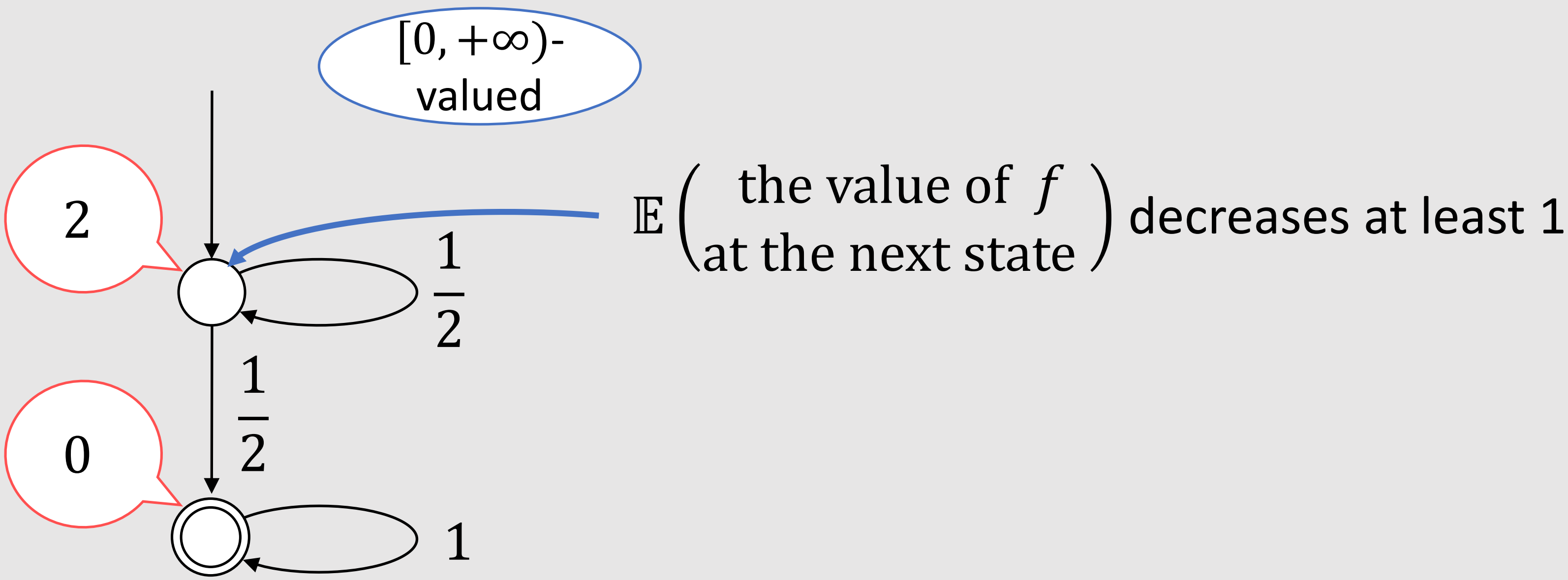


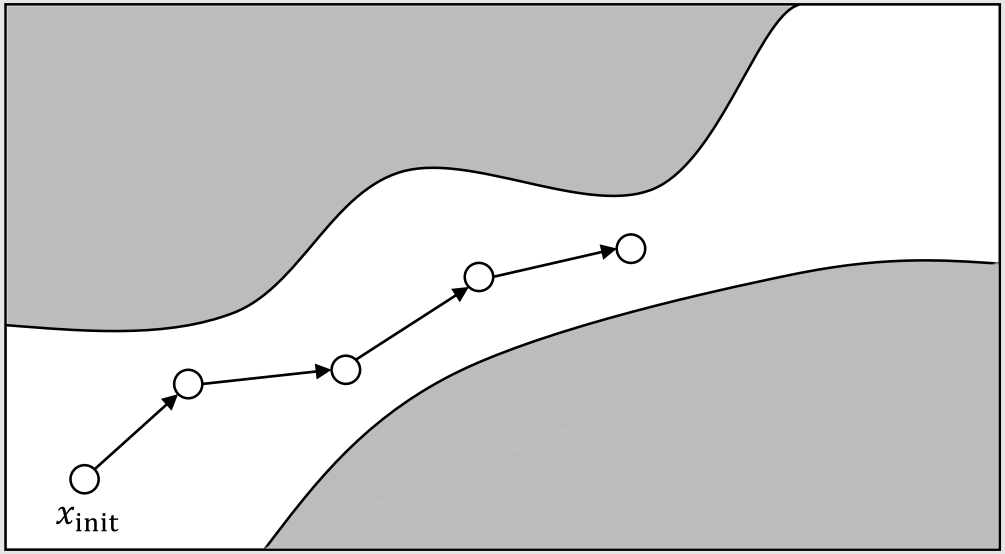
$\mathbb{E}\left(\begin{array}{c}\text{the value of } f \\ \text{at the next state}\end{array}\right)$ decreases at least 1

Ranking supermartingale



The system eventually visits ◎ almost surely

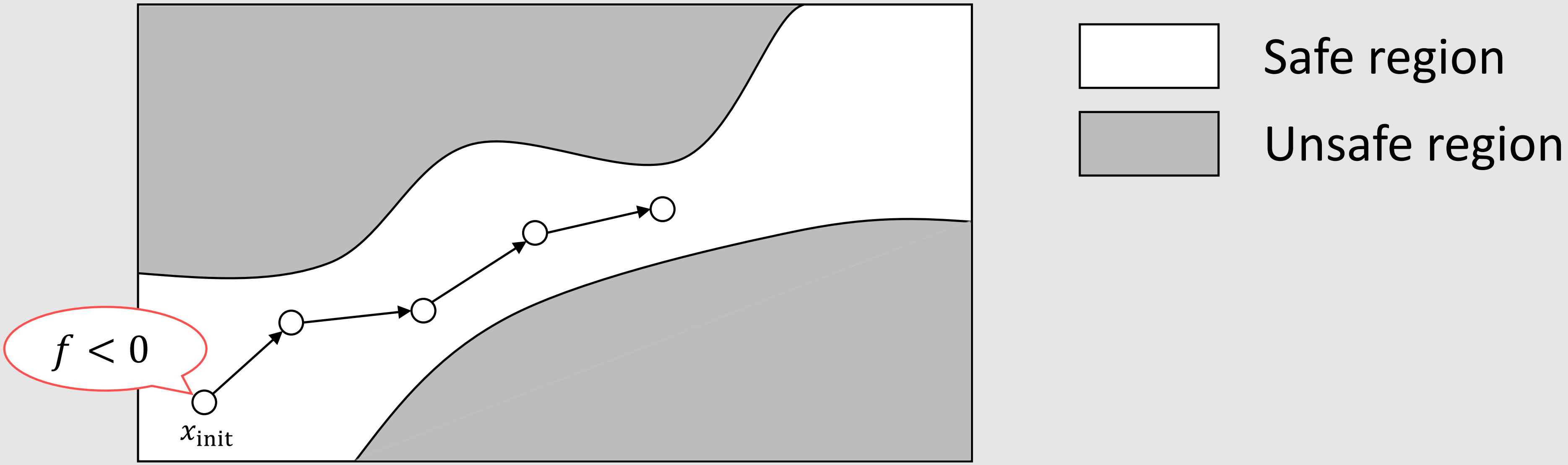
Barrier certificate



Safe region

Unsafe region

Barrier certificate



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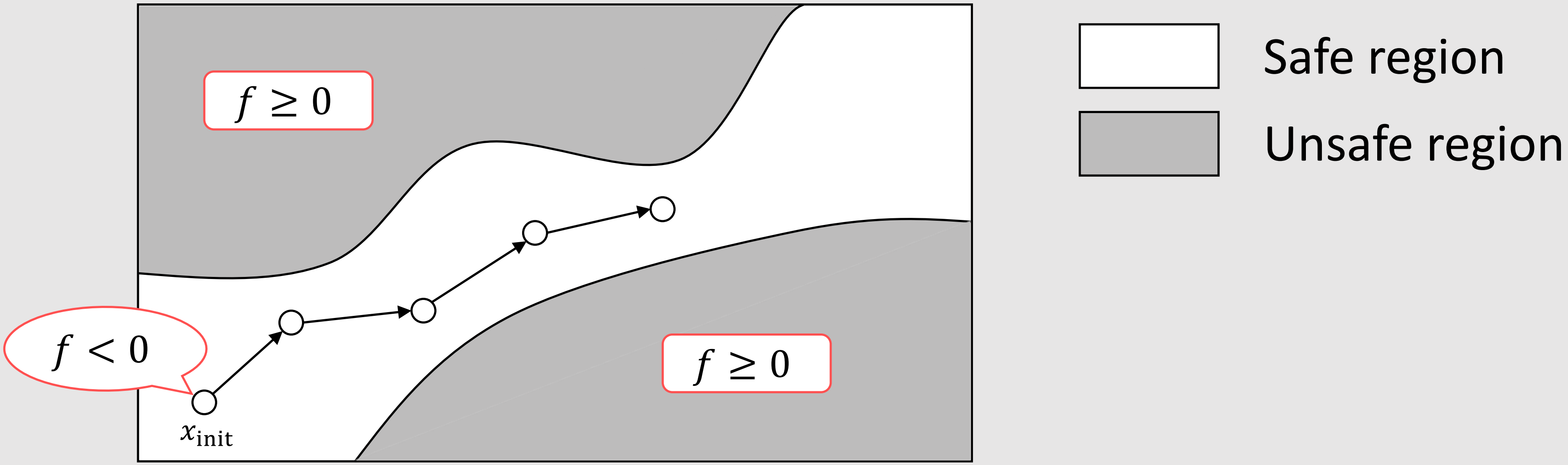

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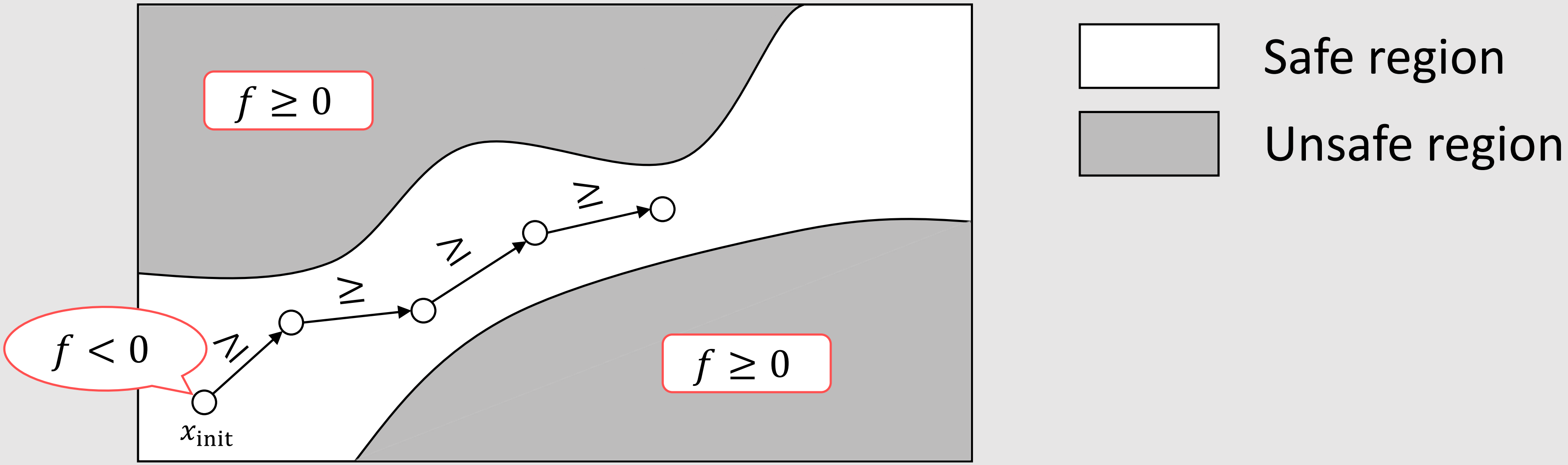

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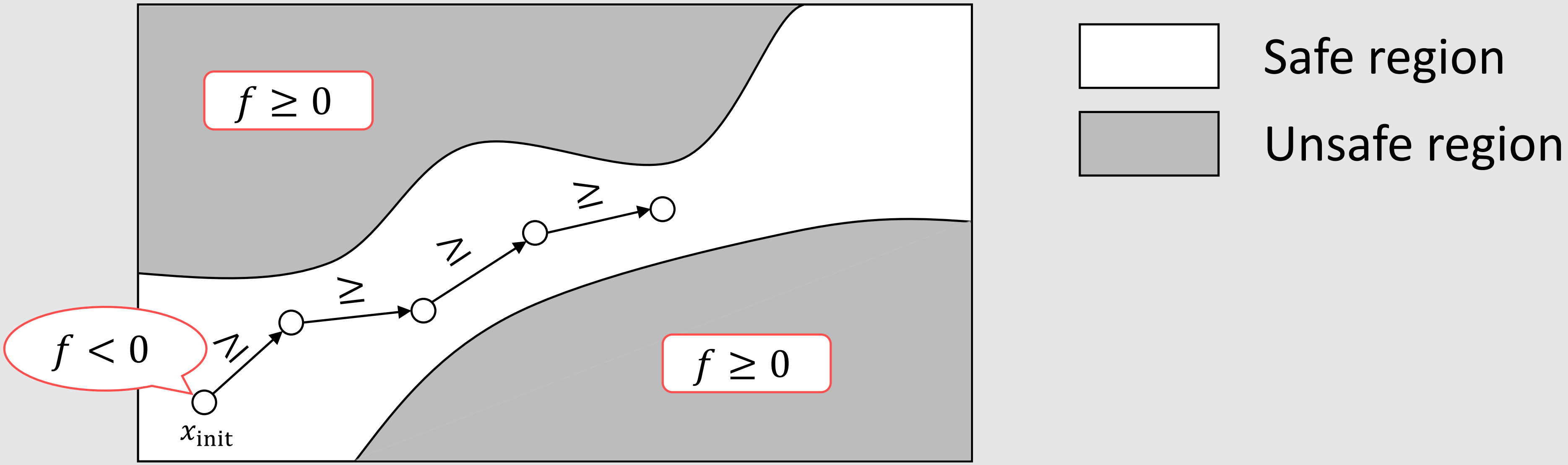
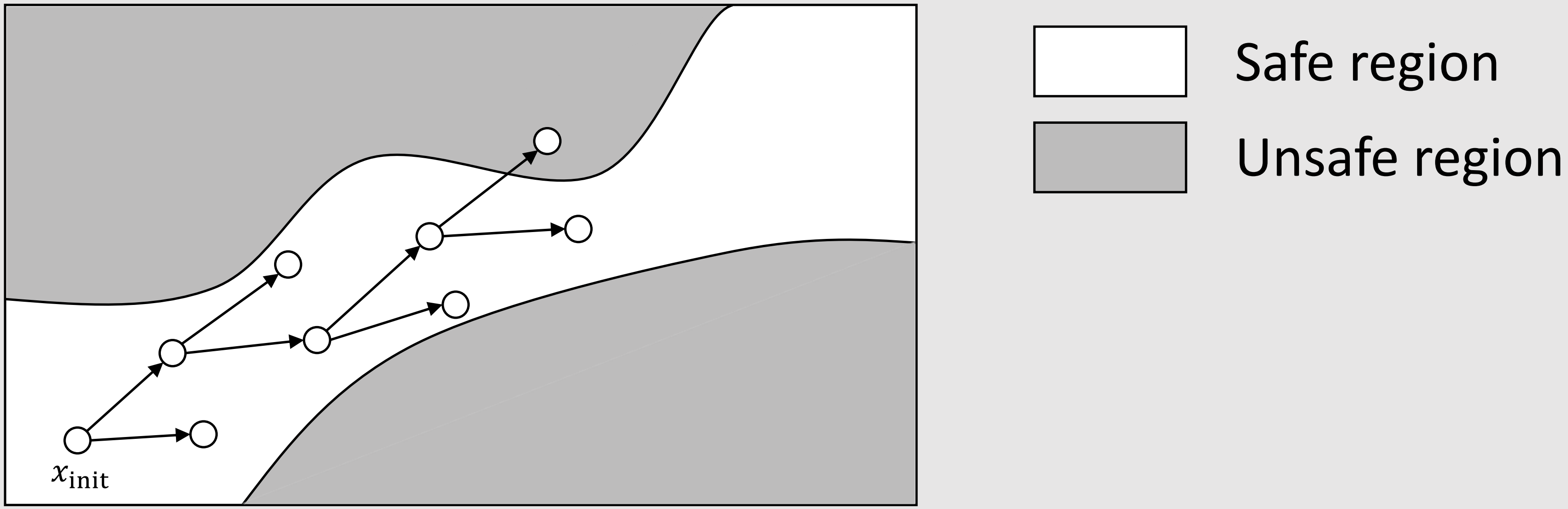


The system does not enter the unsafe region

Probabilistic barrier certificate (a.k.a. nonneg. repulsing supermartingale)



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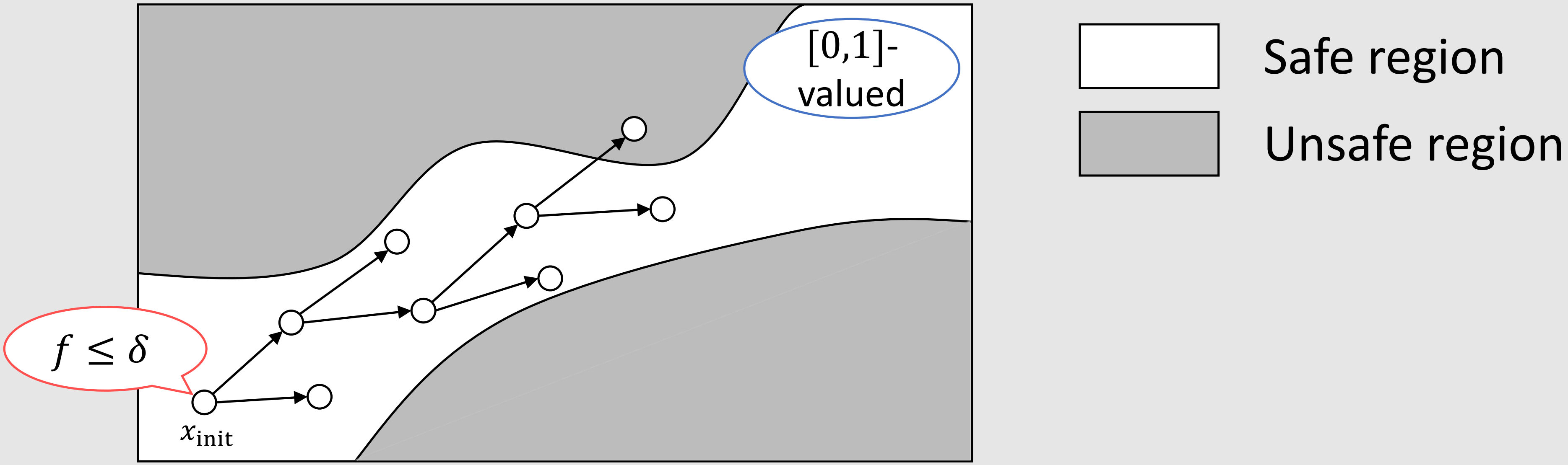

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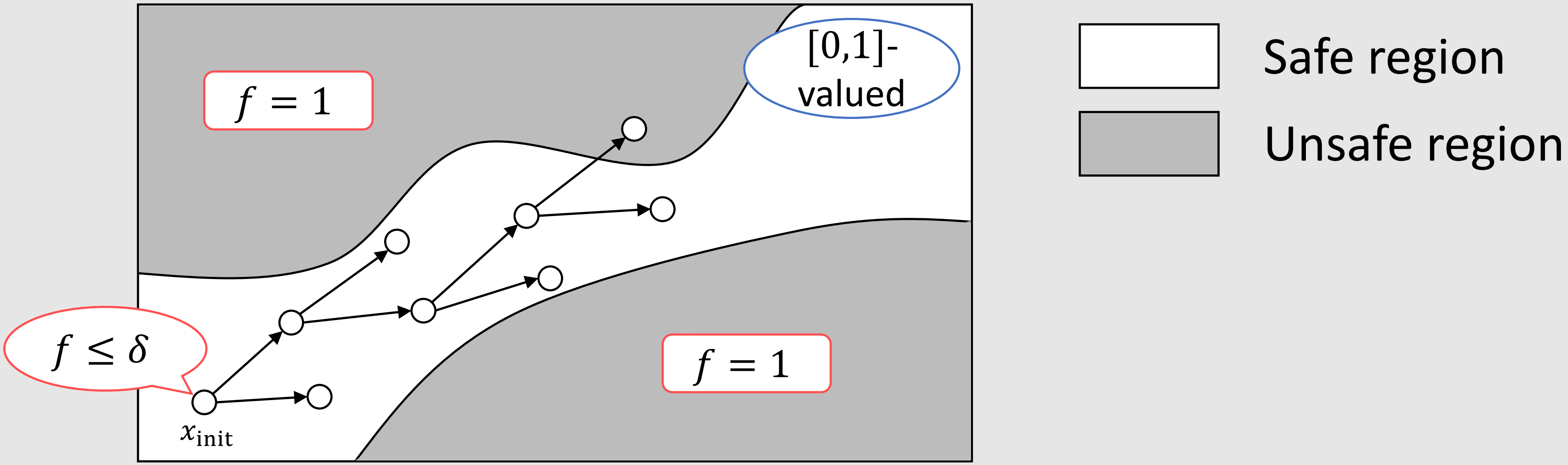

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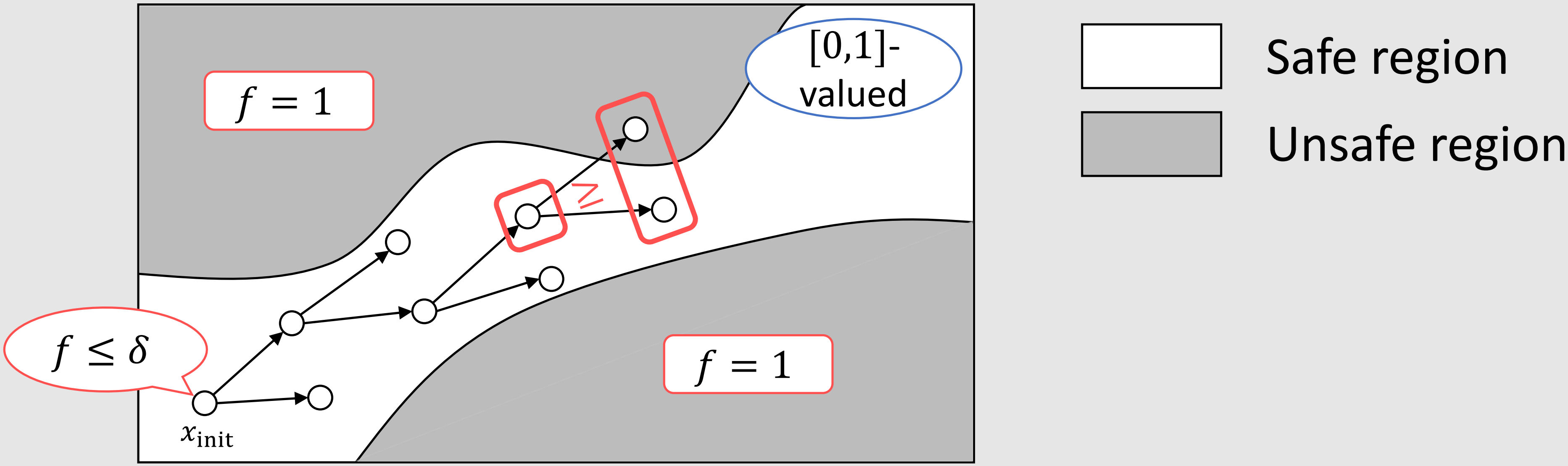

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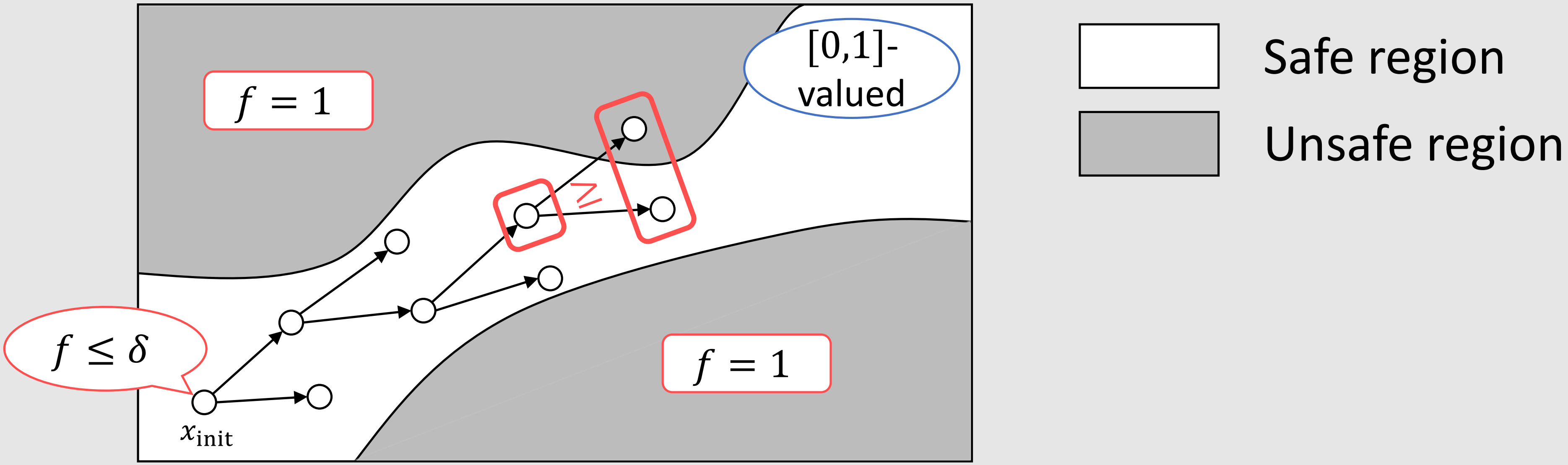

Pr(the system enters the unsafe region) $\leq \delta$

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Two objective functions

- Given: a control flow graph, and a subset C of its states
- For $s \in L \times \mathbb{R}^V =$ (state space),

$$\mathbb{E}^{\text{steps}} : s \mapsto \mathbb{E}\left(\begin{array}{c}\text{the number of steps from } s \\ \text{to the region } C\end{array}\right)$$

$$\mathbb{P}^{\text{reach}} : s \mapsto \mathbb{P}\left(\begin{array}{c}\text{the system eventually visits} \\ \text{the region } C \text{ from } s\end{array}\right)$$

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Soundness/completeness

Ranking supermartingale

Soundness: f is a RankSM $\Rightarrow$ $\mathbb{E}^{\mathrm{steps}} \leq f$

$(f(s) < \infty \Rightarrow \mathbb{P}^{\mathrm{reach}}(s) = 1)$

Completeness: $\mathbb{E}^{\mathrm{steps}}$ is a RankSM

Nonnegative repulsing supermartingale

Soundness: f is a RepSM $\Rightarrow$ $\mathbb{P}^{\mathrm{reach}} \leq f$

Completeness: $\mathbb{P}^{\mathrm{reach}}$ is a RepSM

State of the Art

Approximation method	Certificate for	Soundness	Completeness
Additive ranking Supermartingale (Chakarov-Sankaranarayanan, CAV'13 etc.)	$\mathbb{E}^{\mathrm{steps}} < \infty$ ($\mathbb{P}^{\mathrm{reach}} = 1$)	Yes (with nondet. / continuous variable)	Yes (with nondet. / discrete variable)
Nonnegative repulsing supermartingale (Steinhardt+, IJRR'12 etc.)	$\mathbb{P}^{\mathrm{reach}} \leq ?$	Yes (NO nondet. / continuous variable)*	-
γ-scaled submartingale (Urabe+, LICS'17)	$\mathbb{P}^{\mathrm{reach}} \geq ?$	Yes (NO nondet. / continuous variable)	-
ε-decreasing repulsing supermartingale (Chatterjee+, POPL'17)	$\mathbb{P}^{\mathrm{reach}} \leq ?$	Yes (with nondet. / continuous var. / linearity assumpt.)	-

*In [Steinhardt+] continuous-time dynamics is also considered
Soundness for *c-supermartingale* is shown, which is a relaxation of supermartingale

Our contributions

Soundness/completeness of martingale techniques for PPs with continuous variables and nondeterminism

Characterization of martingale techniques via Knaster-Tarski fixed point theorem

Implementation and experiments

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Soundness/completeness of martingale techniques

Approximation method	Certificate for	Soundness	Completeness
Additive ranking Supermartingale (Chakarov-Sankaranarayanan, CAV'13 etc.)	$\mathbb{E}^{\mathrm{steps}} < \infty$ $(\mathbb{P}^{\mathrm{reach}} = 1)$	Yes (with nondet. / continuous variable)	Yes (with nondet. / cont. var.)
Nonnegative repulsing supermartingale (Steinhardt+, IJRR'12 etc.)	$\mathbb{P}^{\mathrm{reach}} \leq ?$	Yes (with nondet. / continuous variable)*	
γ-scaled submartingale (Urabe+, LICS'17)	$\mathbb{P}^{\mathrm{reach}} \geq ?$	Yes (with nondet. / cont. var.)	-
ε-decreasing repulsing supermartingale (Chatterjee+, POPL'17)	$\mathbb{P}^{\mathrm{reach}} \leq ?$	Yes (with nondet. / continuous var. / linearity assumpt.)	No

*In [Steinhardt+] continuous-time dynamics is also considered
Soundness for *c-supermartingale* is shown, which is a relaxation of supermartingale

Soundness/completeness of martingale techniques

Approximation method	Certificat...	S...ndness	Completeness
Additive rankin... Superm... (Chakar...			Yes ...det. /
No... super... (Steinhardt+, ...			
γ-scaled submartingale (Urabe+, LICS'17)	$\geq ?$	(with nondet. / cont. var.)	-
ε-decreasing repulsing supermartingale (Chatterjee+, POPL'17)	$\mathbb{P}^{\mathrm{reach}} \leq ?$	Yes (with nondet. / continuous var. / linearity assumpt.)	No

For certain endofunctions Φ and Ψ,

$\mathbb{E}^{\mathrm{steps}} = \mu\Phi$ and $\mathbb{P}^{\mathrm{reach}} = \mu\Psi$

*In [Steinhardt+] continuous-time dynamics is also considered
Soundness for *c-supermartingale* is shown, which is a relaxation of supermartingale

Soundness/completeness of RankSM

Our theorem

$$\mathbb{E}^{\text{steps}} = \mu\Phi$$

The lattice $(\mathcal{F}, \sqsubseteq)$

$\mathcal{F}$ ... the set of all (measurable) functions

$$f : L \times \mathbb{R}^V \to [0, \infty]$$

$\sqsubseteq$ ... $f \sqsubseteq g \iff \forall s.\, f(s) \le g(s)$

Soundness/completeness of RankSM

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Soundness

$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\text{steps}} \sqsubseteq f}$$

Soundness/completeness of RankSM

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$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s. f(s) \leq g(s)$

Soundness

$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\text{steps}} \sqsubseteq f} \quad \Leftrightarrow \quad \frac{\Phi f \sqsubseteq f}{\mu\Phi \sqsubseteq f}$$

Soundness/completeness of RankSM

Our theorem

$$\mathbb{E}^{\text{steps}} = \mu\Phi$$

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$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\text{steps}} \sqsubseteq f} \quad \begin{matrix} \Leftrightarrow \\ \Leftrightarrow \end{matrix} \quad \boxed{\frac{\Phi f \sqsubseteq f}{\mu\Phi \sqsubseteq f}}$$

Knaster-Tarski theorem

Soundness/completeness of RankSM

Our theorem

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$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s.\, f(s) \leq g(s)$

Soundness

$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\text{steps}} \sqsubseteq f} \quad \begin{array}{c}\Leftrightarrow \\ \Leftrightarrow\end{array} \quad \boxed{\frac{\Phi f \sqsubseteq f}{\mu\Phi \sqsubseteq f}}$$

Knaster-Tarski theorem

Completeness

$$\Phi\mathbb{E}^{\text{steps}} \sqsubseteq \mathbb{E}^{\text{steps}}$$

Soundness/completeness of NNRepSM

Our theorem

$$\mathbb{P}^{\mathrm{reach}} = \mu\Psi$$

The lattice $(\mathcal{F}, \sqsubseteq)$

$\mathcal{F}$ ... the set of all (measurable) functions

$$f : L \times \mathbb{R}^V \to [0,1]$$

$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s. f(s) \leq g(s)$

Soundness

$$\frac{f \text{ is a RepSM}}{\mathbb{P}^{\mathrm{reach}} \sqsubseteq f} \Leftrightarrow \boxed{\frac{\Psi f \sqsubseteq f}{\mu\Psi \sqsubseteq f}}$$

Knaster-Tarski theorem

Completeness

$$\Psi\mathbb{P}^{\mathrm{reach}} \sqsubseteq \mathbb{P}^{\mathrm{reach}}$$

Our contributions

Soundness/completeness of martingale techniques for PPs with continuous variables and nondeterminism

Characterization of martingale techniques via Knaster-Tarski fixed point theorem

Implementation and experiments

Synthesis algorithm

Approximation method	Certificate for	Soundness	Completeness
Additive ranking Supermartingale (Chakarov-Sankaranarayanan, CAV'13 etc.)	$\mathbb{E}^{\mathrm{steps}} < \infty$ ($\mathbb{P}^{\mathrm{reach}} = 1$)	Yes (with nondet. / continuous variable)	Yes (with nondet. / cont. var.)
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- Input: affine/polynomial PP
- Translate PP to
  - control flow graph
  - initial state x_{init}
  - set of terminal states
- For the set F of all affine/polynomial functions over states, solve:

$$\operatorname*{minimize}_{f \in F} f(x_{init}) \quad , \quad \text{subject to (Upper NNRepSupM condition)}$$

- Output: $f(x_{init})$

Overapprox. " $\sup \mathbb{P}^{\mathrm{reach}}$ "

Approximation method	Certificate for	Soundness	Completeness
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Synthesis algorithm

- Input: affine/polynomial PP
- Translate PP to
  - control flow graph
  - initial state x_{init}
  - set of terminal states

Can be reduced to LP/SDP problem
(e.g. Chakarov-Sankaranarayanan, CAV'13; Chatterjee+, CAV'16)

- For the set F of all affine/polynomial functions over states, solve:

$$\underset{f \in F}{\text{minimize}}\ f(x_{init}) \quad , \quad \text{subject to (Upper NNRepSupM condition)}$$

- Output: $f(x_{init})$

Overapprox. “ $\sup \mathbb{P}^{\mathrm{reach}}$ ”

Synthesis algorithm

Approximation method	Certificate for	Soundness	Completeness
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- Input: affine/polynomial PP
- Translate PP to
  - control flow graph
  - initial state x_{init}
  - set of terminal states

Can be reduced to LP problem
(e.g. Chakarov-Sankaranarayanan, CAV'13)

- For the set F of all affine functions over states, solve:

$$\max_{f \in F} f(x_{init}) \quad , \quad \text{subject to (Lower } \gamma\text{-SclSubM condition)}$$

- Output: $f(x_{init})$

Underapprox. " $\inf \mathbb{P}^{\mathrm{reach}}$ "

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Experiments

γ-scaled submartingale

		Prog. III (linear)	
	param.	time (s)	bound
(a-1)	$p_1 = 0.2$ $p_2 = 0.4$	0.026	≥ 0
	$p_1 = 0.8$ $p_2 = 0.1$	0.022	≥ 0.751
(a-2)	$M_1 = -1$ $M_2 = 2$	0.033	≥ 0
	$M_1 = -2$ $M_2 = 1$	0.033	≥ 0.767
(a-3)	$M_1 = -1$ $M_2 = 2$	0.028	≥ 0
	$M_1 = -2$ $M_2 = 1$	0.040	≥ 0.801
(b)	$c = 0.1$ $p = 0.5$	0.056	≥ 0
	$c = 0.1$ $p = 0.1$	0.054	≥ 0.148

Nonnegative repulsing submartingale

		Prog. I (linear)		Prog. II (deg.-2 poly.)		Prog. II (deg.-3 poly.)	
	param.	time (s)	bound	time (s)	bound	time (s)	bound
(a-1)	$p_1 = 0.2$ $p_2 = 0.4$	0.021	≤ 0.825	530.298	≤ 0.6552	572.393	≤ 0.6555
	$p_1 = 0.8$ $p_2 = 0.1$	0.024	≤ 1	526.519	≤ 1.0	561.327	≤ 1.0

- Input: adversarial random walk (similar to the reading ex.)
- Nontrivial bounds found in 50% cases

Experiments

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	true reachability probability	U-NNRepSupM	1-RepSupM
(c-1)	$\frac{(0.4/0.6)^5 - (0.4/0.6)^{10}}{1-(0.4/0.6)^{10}} \approx 0.116$	0.505	< 1
(c-2)	0.5	0.5	—
(c-3)	$\int_0^1 \left(\frac{0.25}{0.75}\right)^{\lceil \log_2(1/x) \rceil} dx \approx 0.2$	0.5	—
(c-4)	$\left(\frac{0.25}{0.75}\right)^1 \approx 0.333$	—	< 1

Observed comparative advantage of nonnegative RepSM over ε-decreasing RepSM

Thank you for your attention ☺

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