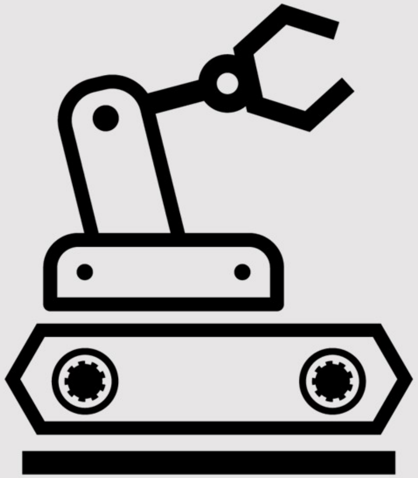
Ranking and Repulsing Supermartingales for Reachability in Probabilistic Programs

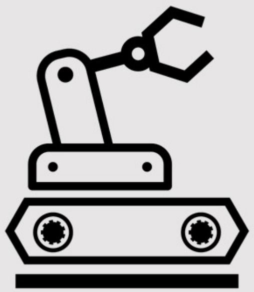
Toru Takisaka, Yuichiro Oyabu, Natsuki Urabe, Ichiro Hasuo

ERATO 蓮尾メタ数理システムデザインプロジェクト
ERATO Metamathematics for Systems Design Project
国立情報学研究所 & 科学技術振興機構 National Institute of Informatics & Japan Science and Technology Agency



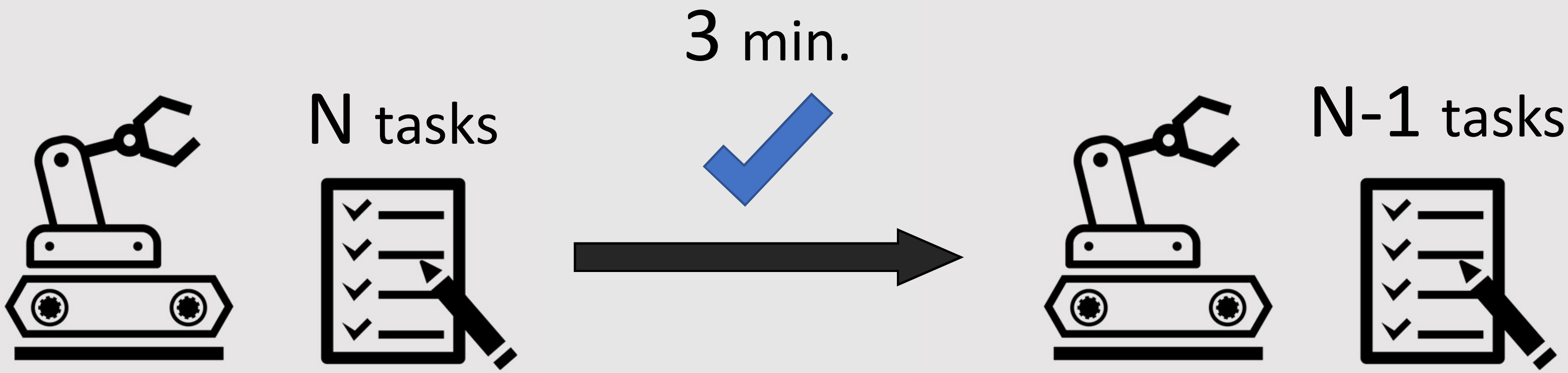
A robot resolves a set of tasks

Mode 1: safe mode



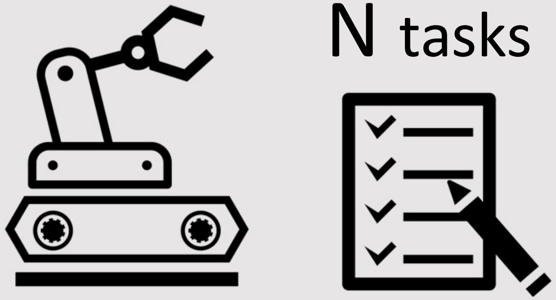
N tasks

Mode 1: safe mode



Mode 2: urgent mode

N tasks



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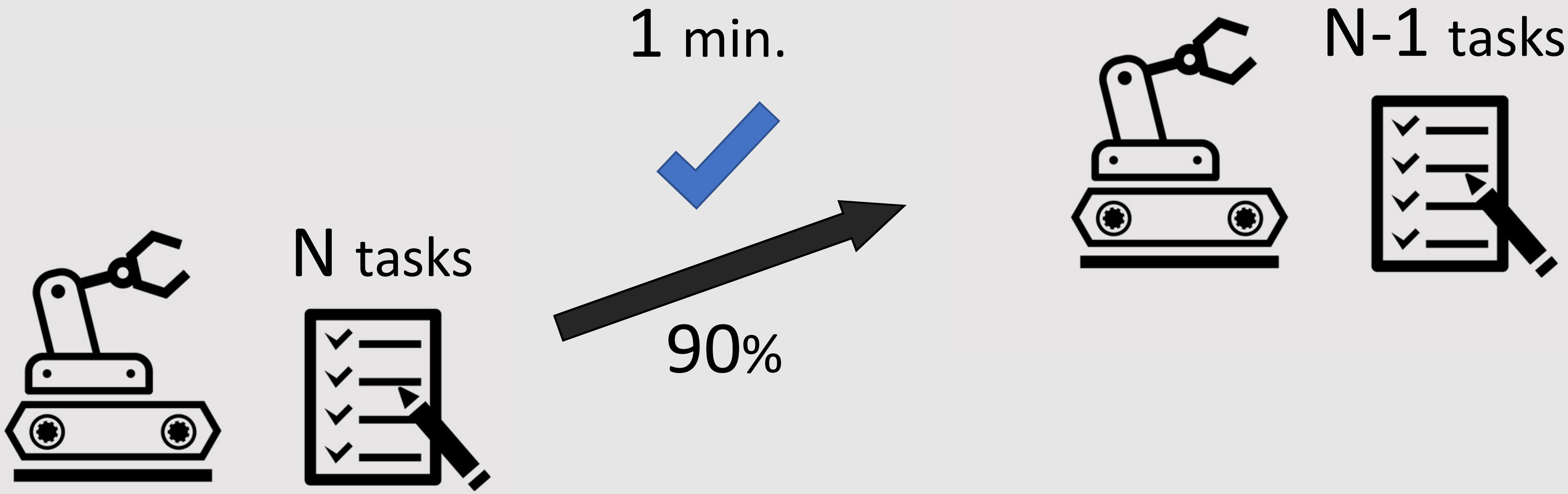

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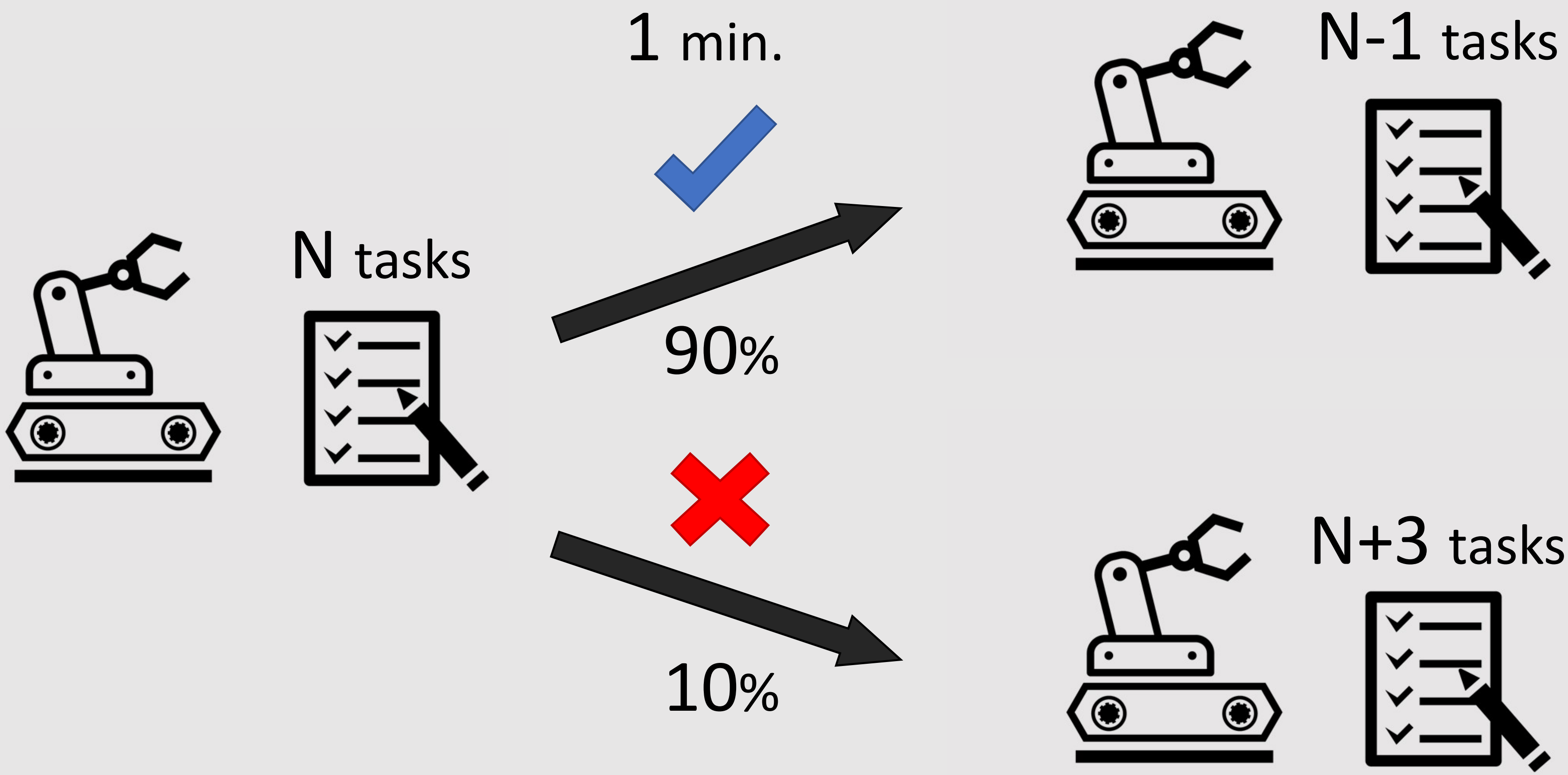
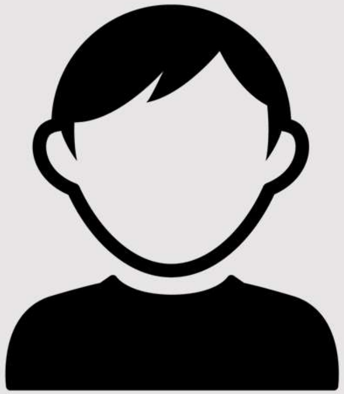
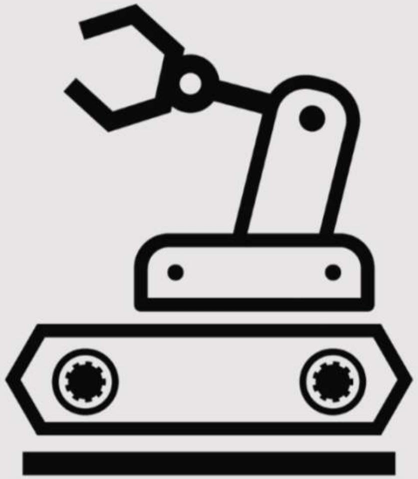
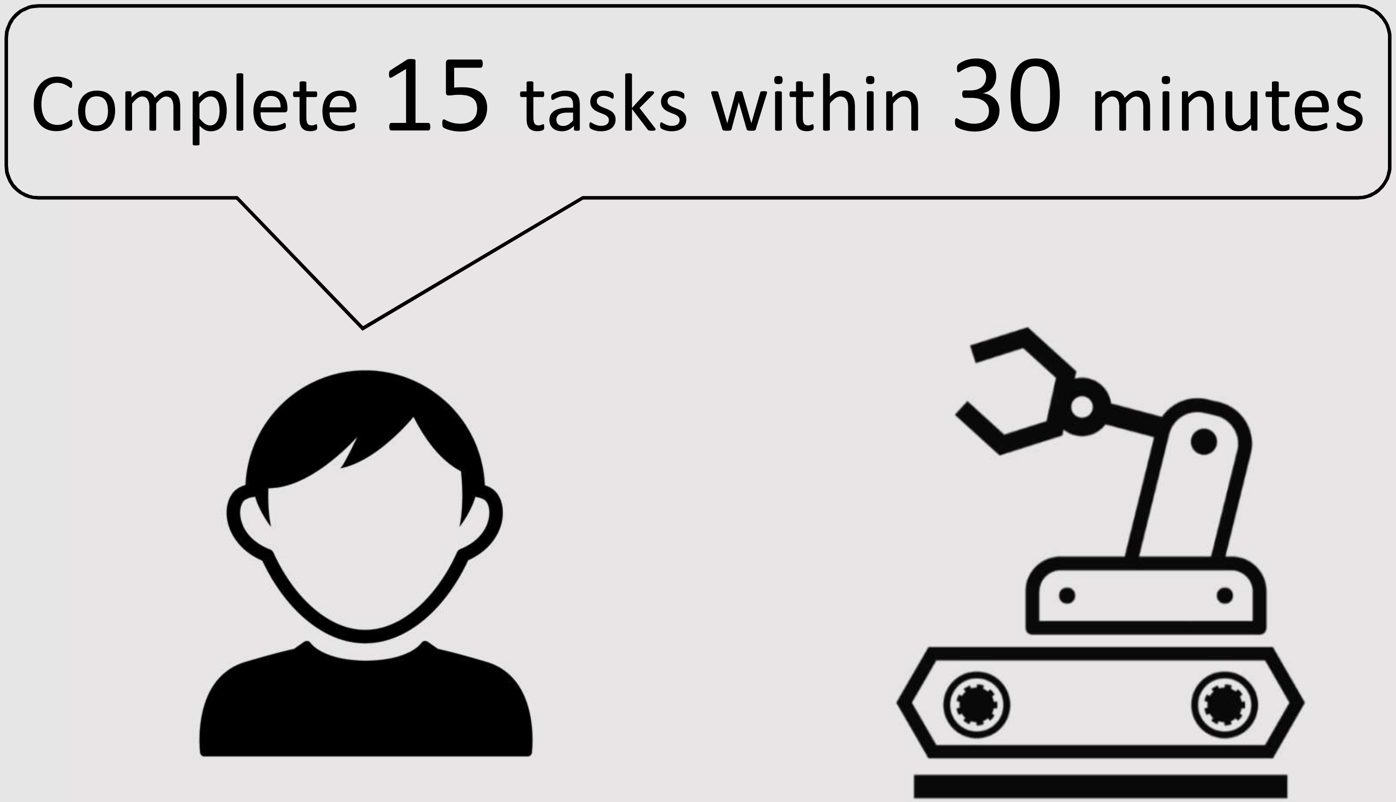
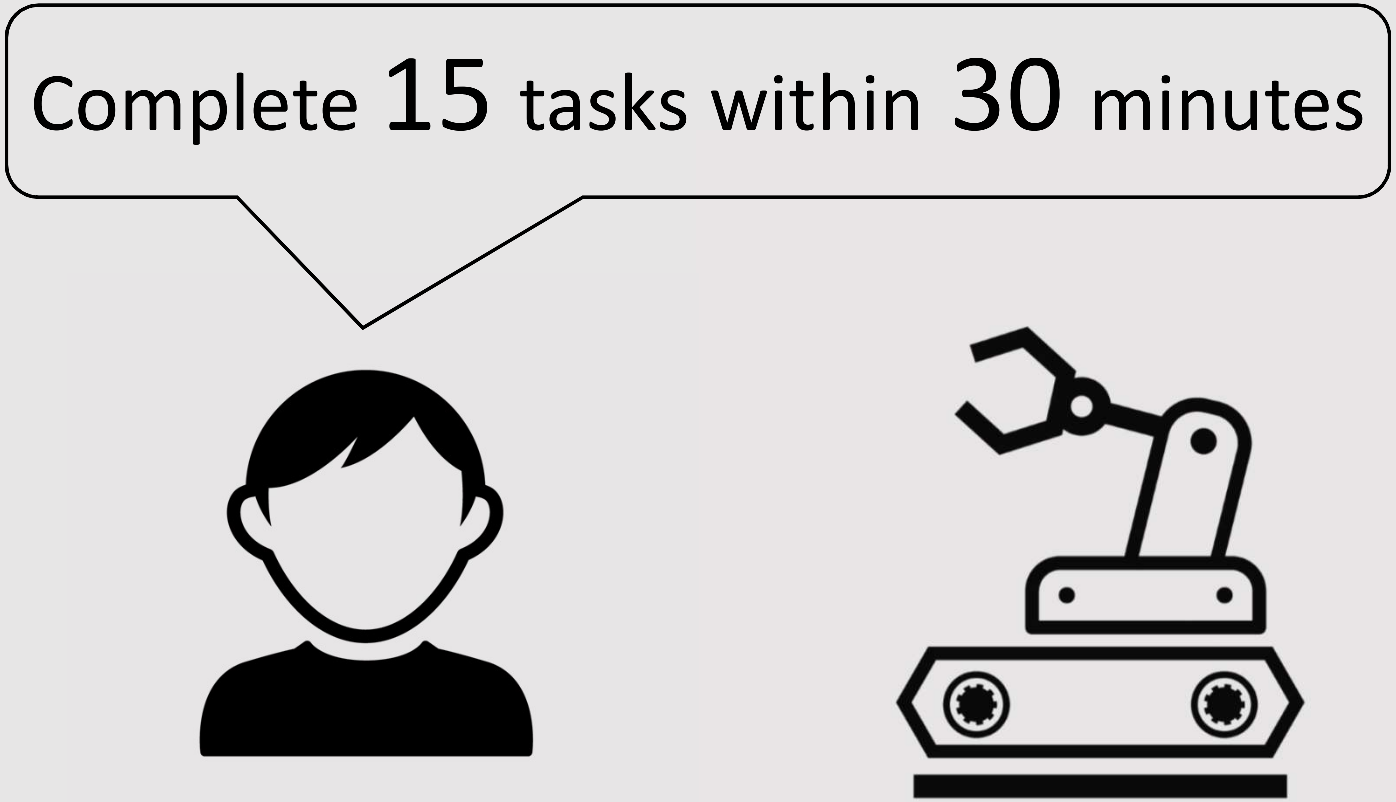







Complete 15 tasks within 30 minutes



What is the probability that
the robot completes the tasks?

Problem formulation

Input: probabilistic program

```
x := 15; t := 0;
p := {0.9:1, 0.1:-3};
while x > 0 do
  if * then
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refute (t > 30)
```

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Nondet. / Prob. branching

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Problem

What is the probability that the program terminates?

(under angelic/demonic scheduler)

We admit continuous variable

⇒Generally one can't compute this value efficiently

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⇒ Certification by supermartingale

Certification by supermartingale

Probabilistic modification of real-world benchmarks

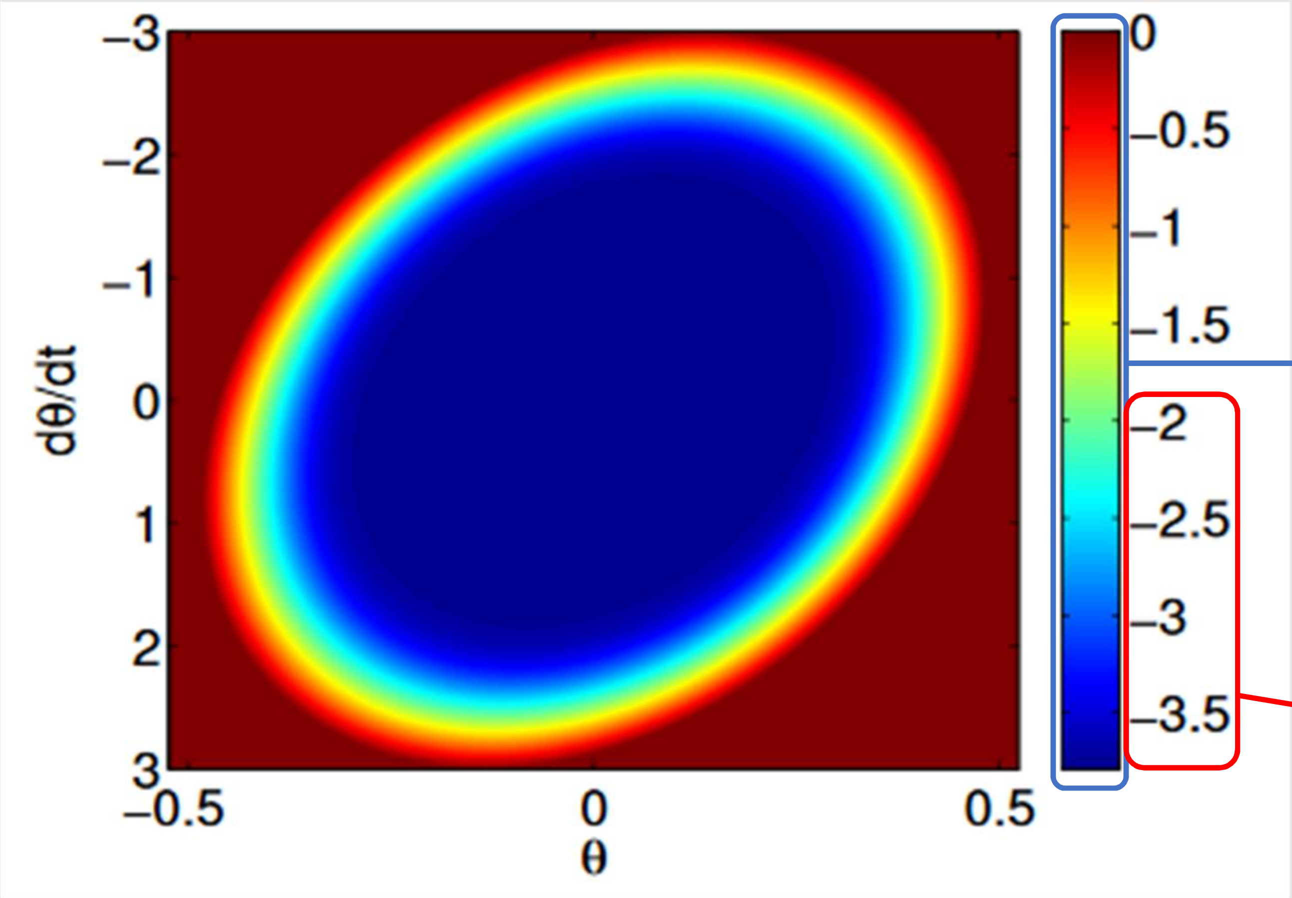
(in Alias+, SAS'10)

Almost-sure termination is certified in 20/28 examples

(Agrawal+, POPL'18)

Benchmark	Time (s)	Solution	Dimension	Prob. loops	Prob. Assignments
alain	0.11	yes	2	yes	yes
catmouse	0.08	yes	2	yes	yes
counterex1a	0.1	no		no	no
counterex1c	0.11	yes	3	yes	yes
easy1	0.09	yes	1	yes	yes
exmini	0.09	yes	2	yes	yes
insertsort	0.1	yes	3	yes	yes
ndecr	0.09	yes	2	yes	yes
perfect	0.11	yes	3	yes	yes
perfect2	0.1	yes	3	yes	no
	0.11	no		yes	yes
real2	0.09	no		no	no
realbubble	0.22	yes	3	yes	yes
realselect	0.11	yes	3	yes	yes
realshellsort	0.09	no		yes	no
serpent	0.1	yes	1	yes	yes
sipmabubble	0.1	yes	3	yes	yes
speedDis2	0.09	no		no	no
speedNestedMultiple	0.1	yes	3	yes	yes
speedpldi2	0.09	yes	2	yes	yes
speedpldi4	0.09	yes	3	yes	yes
speedSimpleMultipleDep	0.09	no		no	no
speedSingleSingle2	0.12	yes	2	yes	no
	0.1	no		yes	yes
unperfect	0.1	yes	2	yes	no
	0.16	no		yes	yes
wcet1	0.11	yes	2	yes	yes
while2	0.1	yes	3	yes	yes

Certification by supermartingale



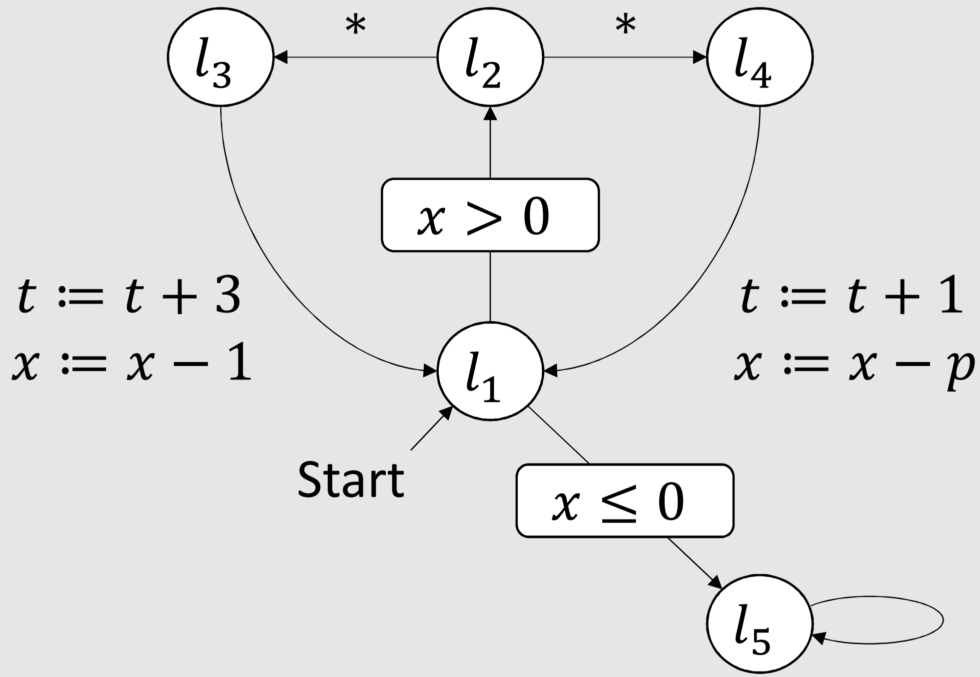
System: a pendulum under Gaussian noise

The log-base-10 of the failure probability
(failure = $|\theta| > \pi/6$ within 1h)

>99% safety is guaranteed
(Pr(enter a bad state) <1%)

(Steinhardt-Tedrake, IJRR'12)

Control flow graph

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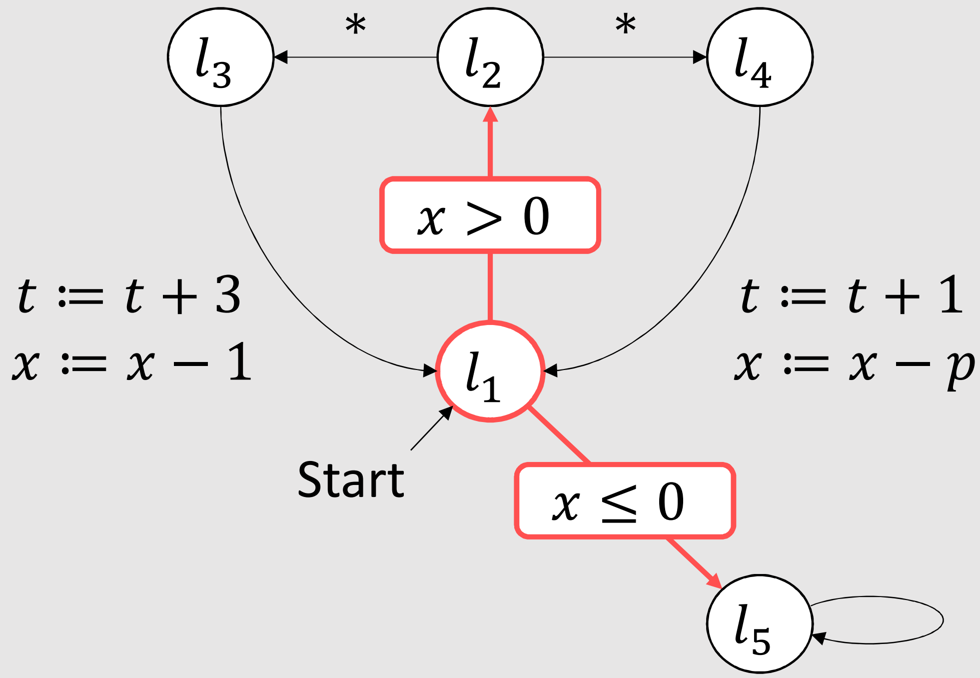
- As powerful as MDP

finite

$\mathbb{R}^V$

Control flow graph

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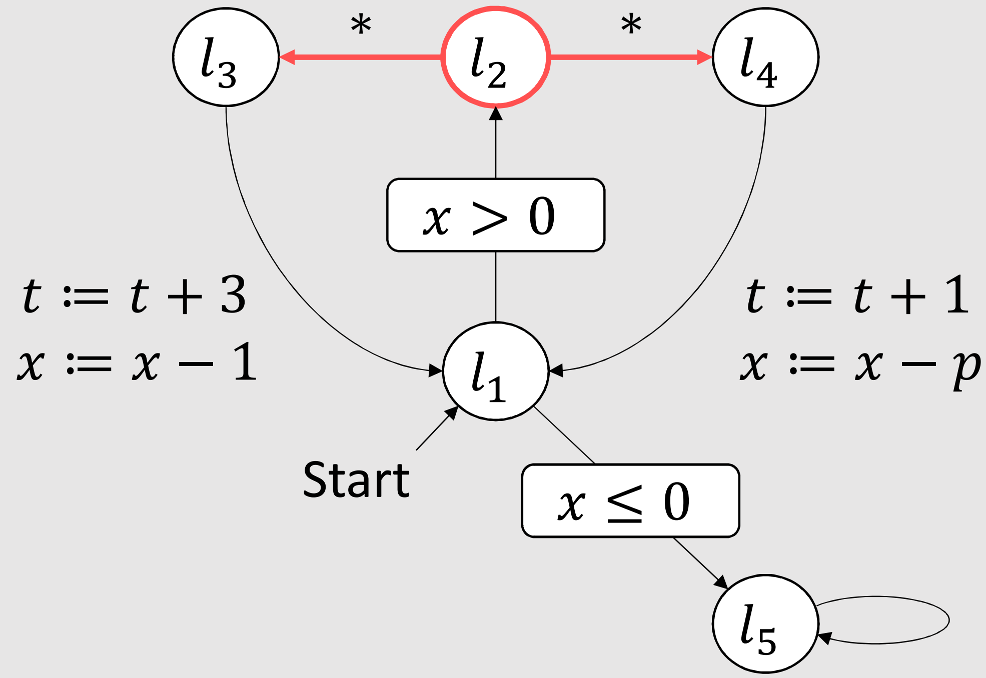


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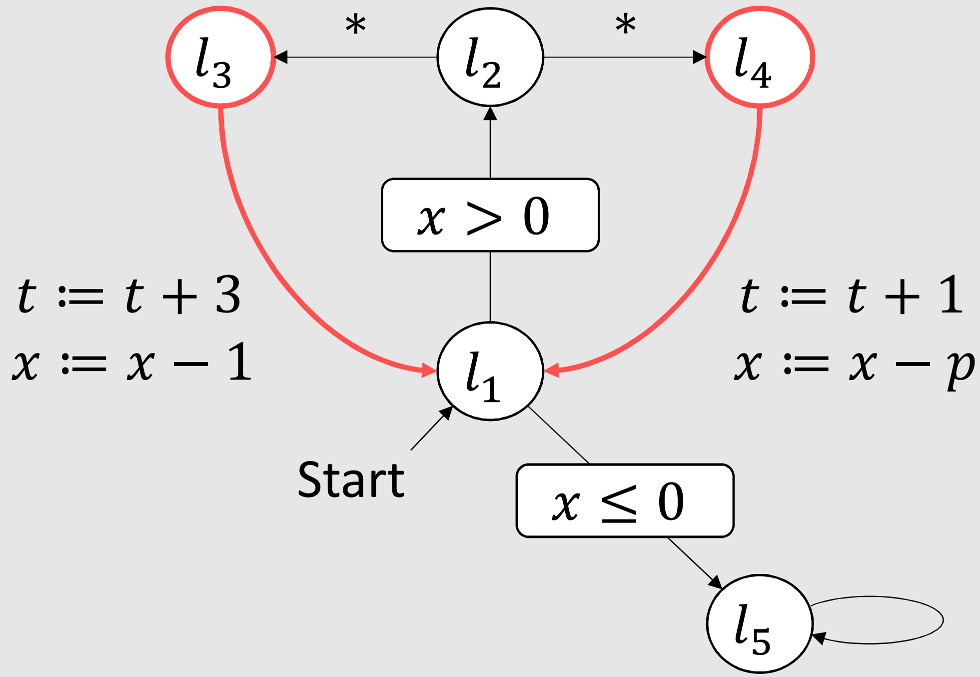


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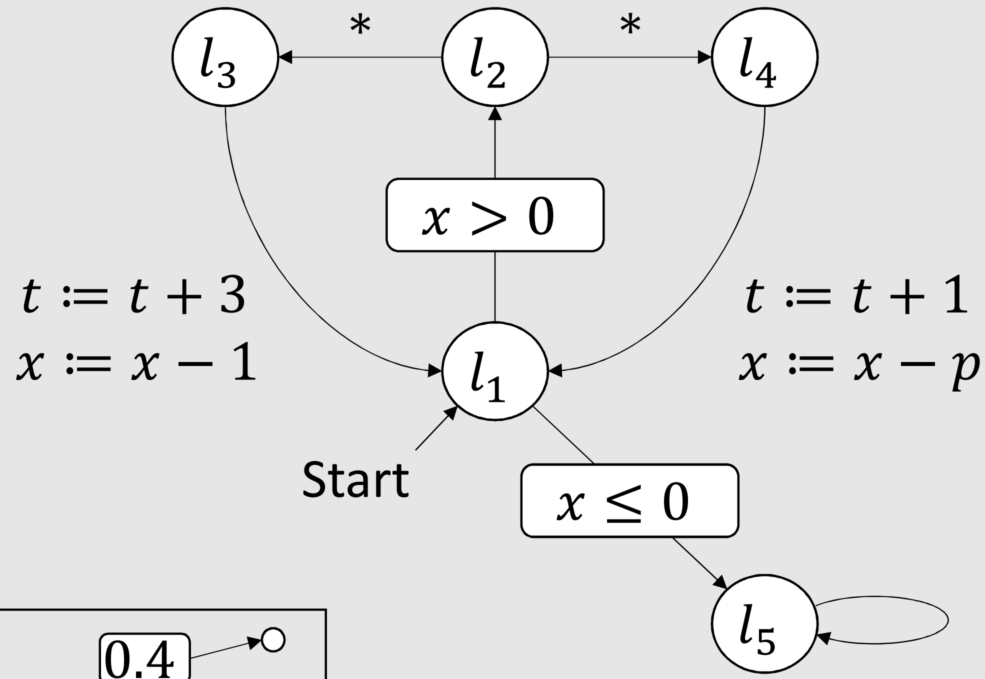


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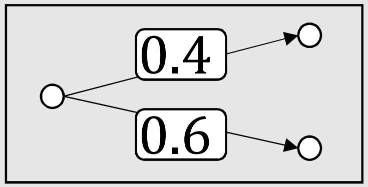

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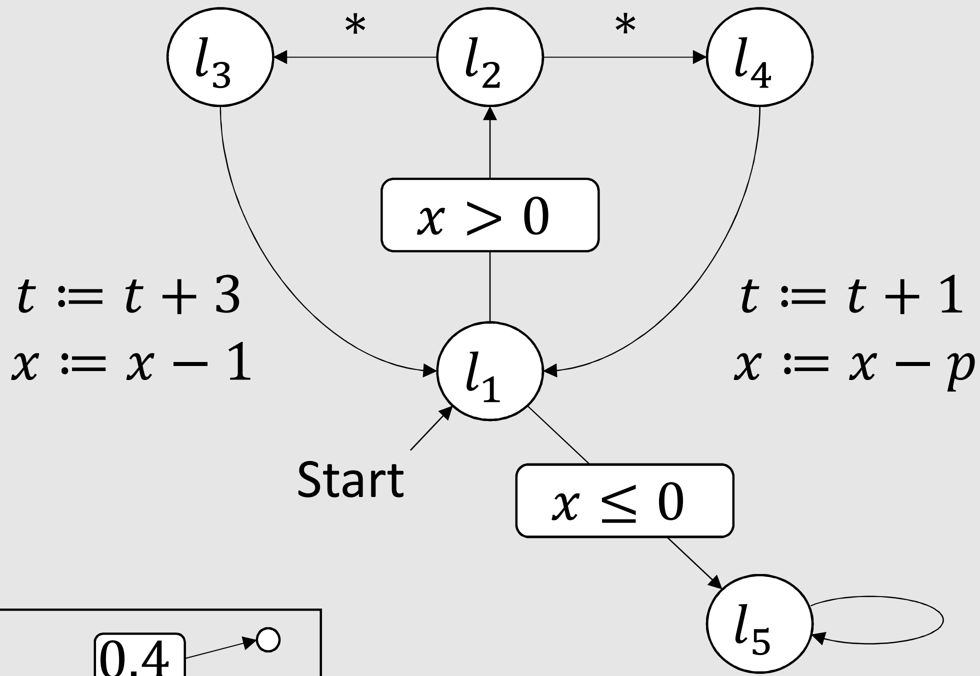


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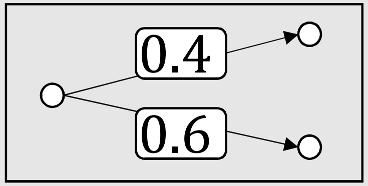

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$\mathbb{R}^V$

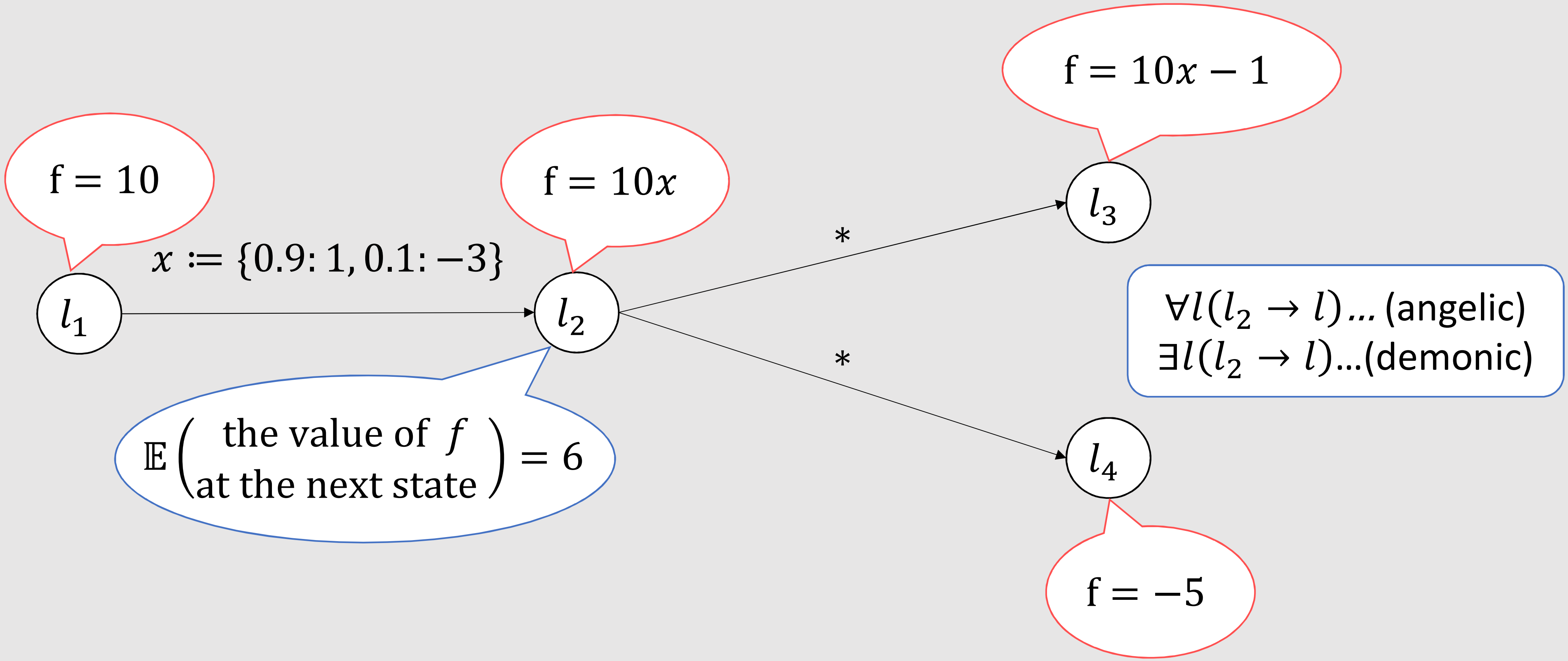
Problem

$$C \coloneqq \{l_5\} \times (\mathbb{R} \times (30, \infty))$$

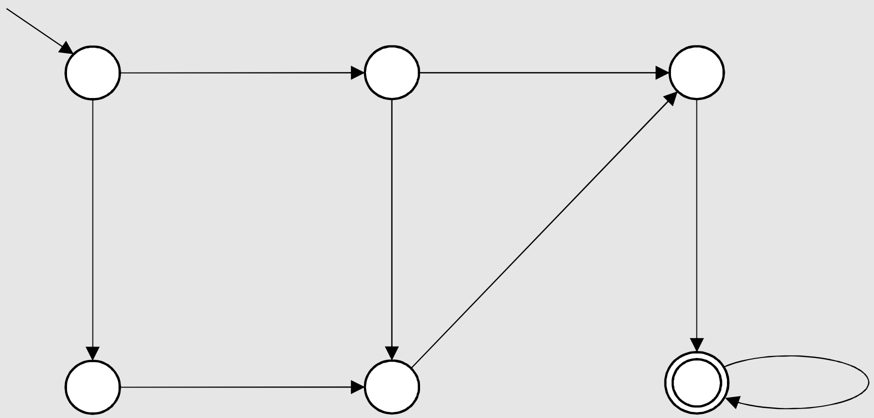
$$\subseteq (\text{Locations}) \times (\text{Variables})$$

⇒Pr(the system eventually visits the region C)?

Supermartingale = a function over states that is "non-increasing" through transitions



Ranking function



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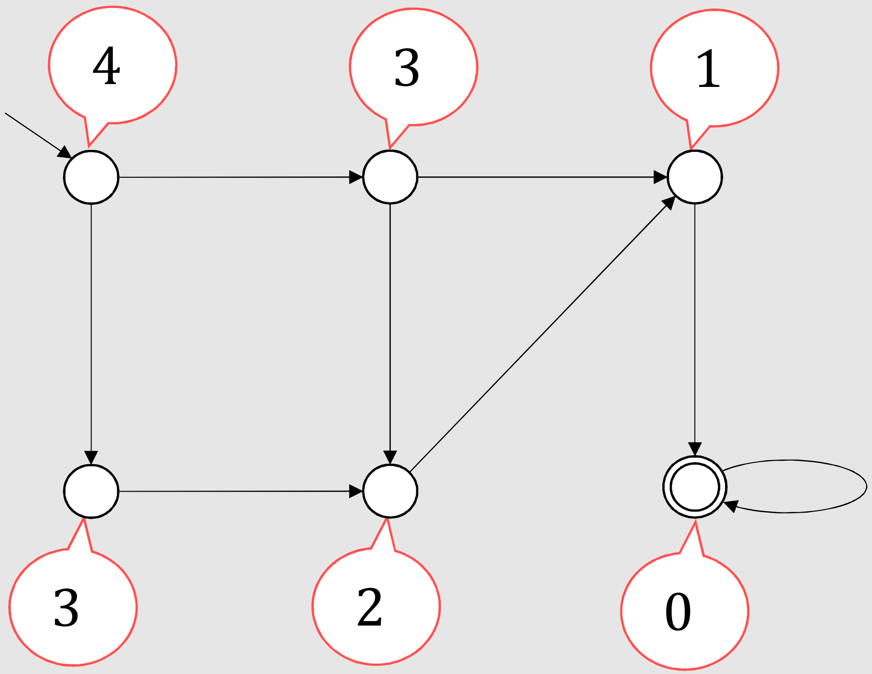
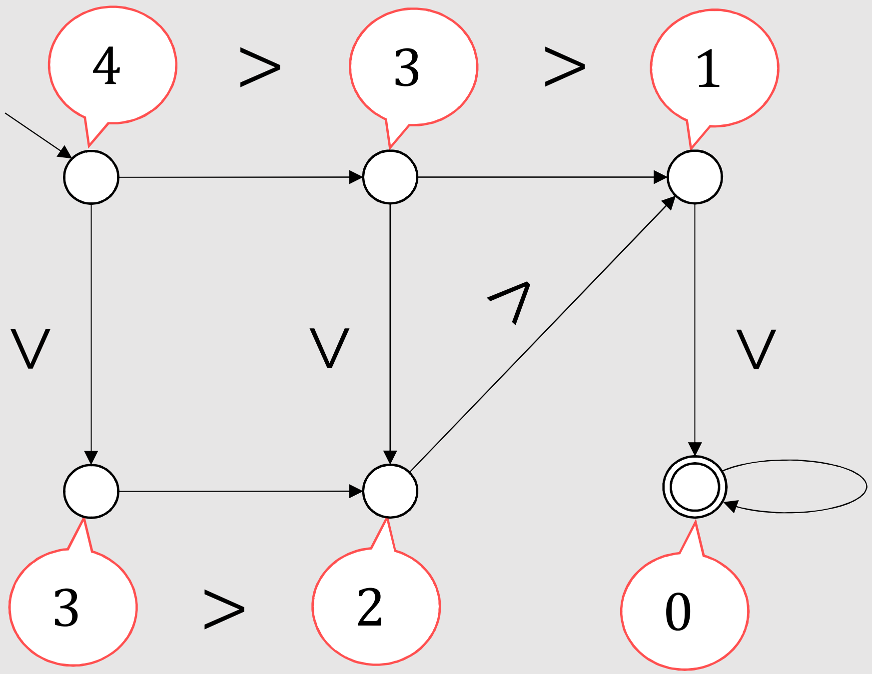


Ranking function

Int-valued



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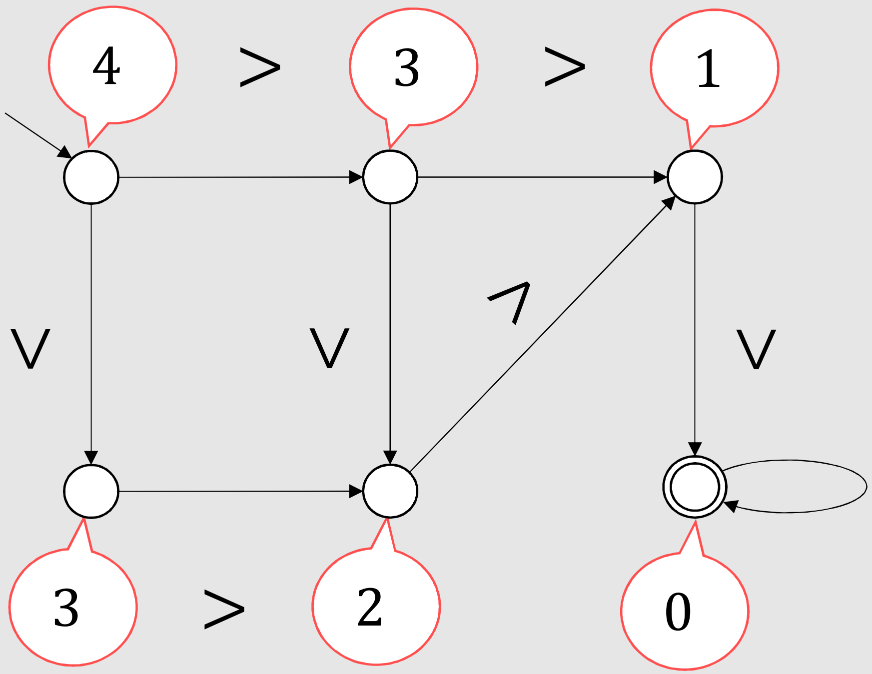

The system eventually visits ◎ (under any nondeterministic choice)

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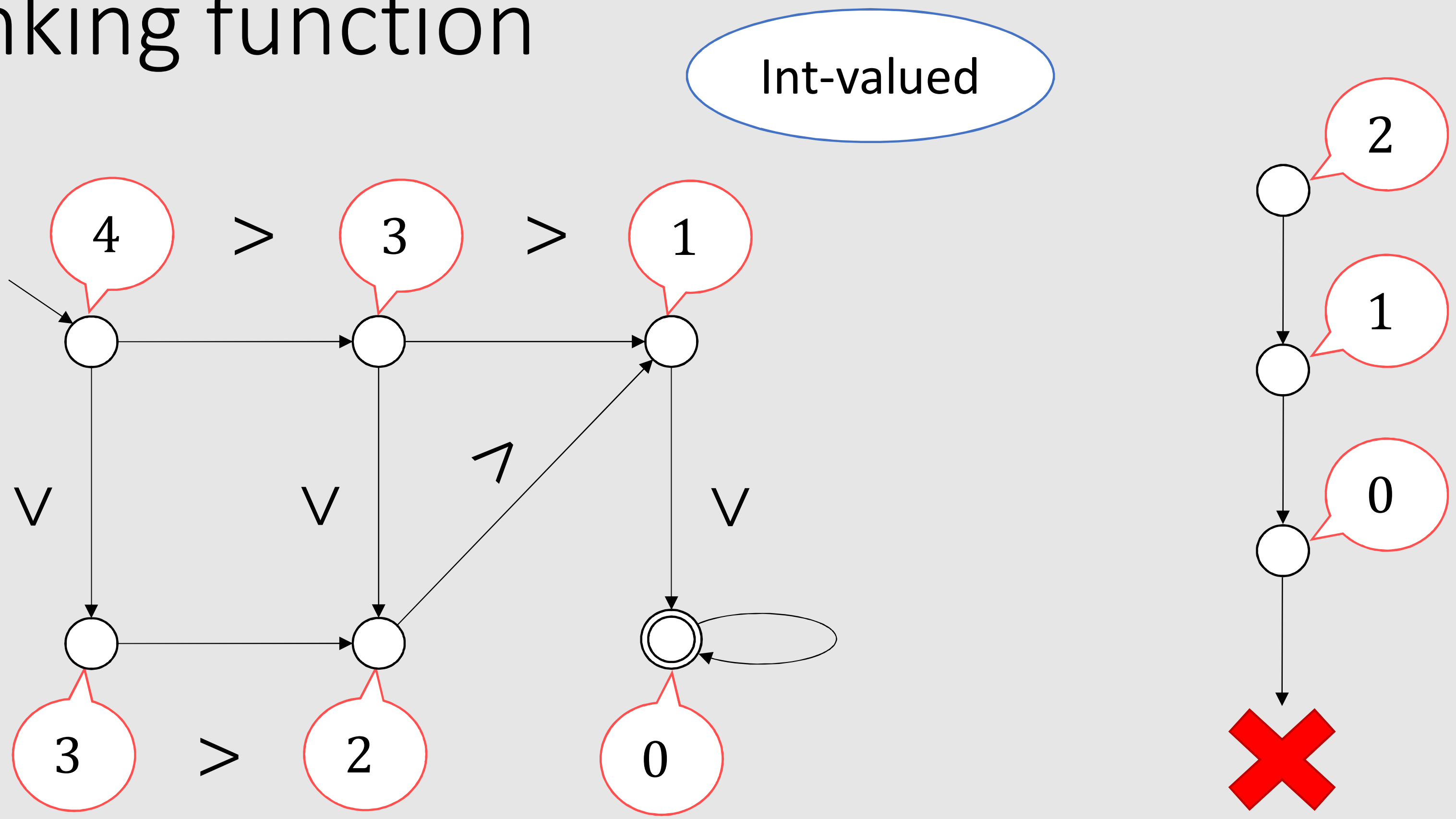

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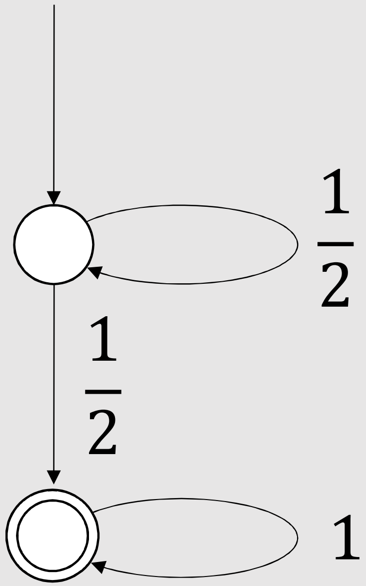
Ranking supermartingale



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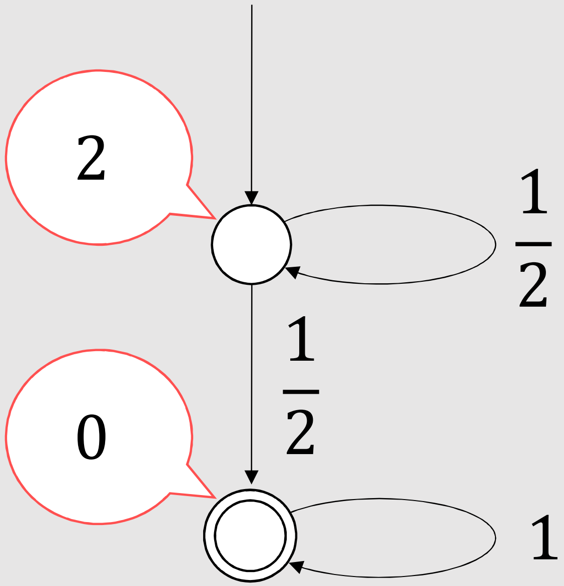

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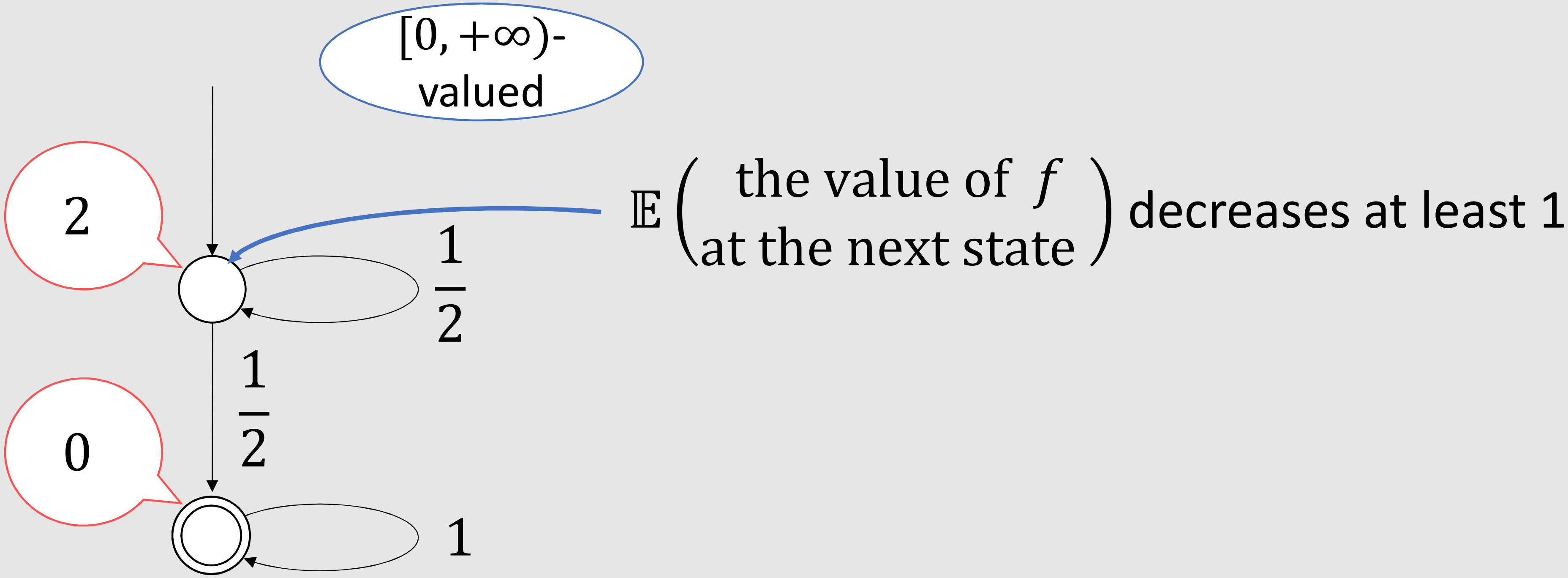
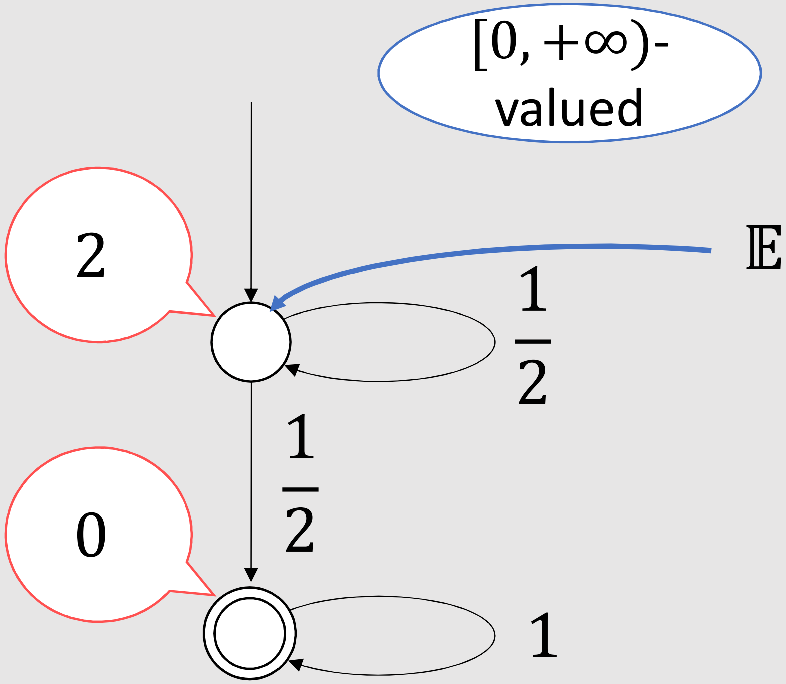


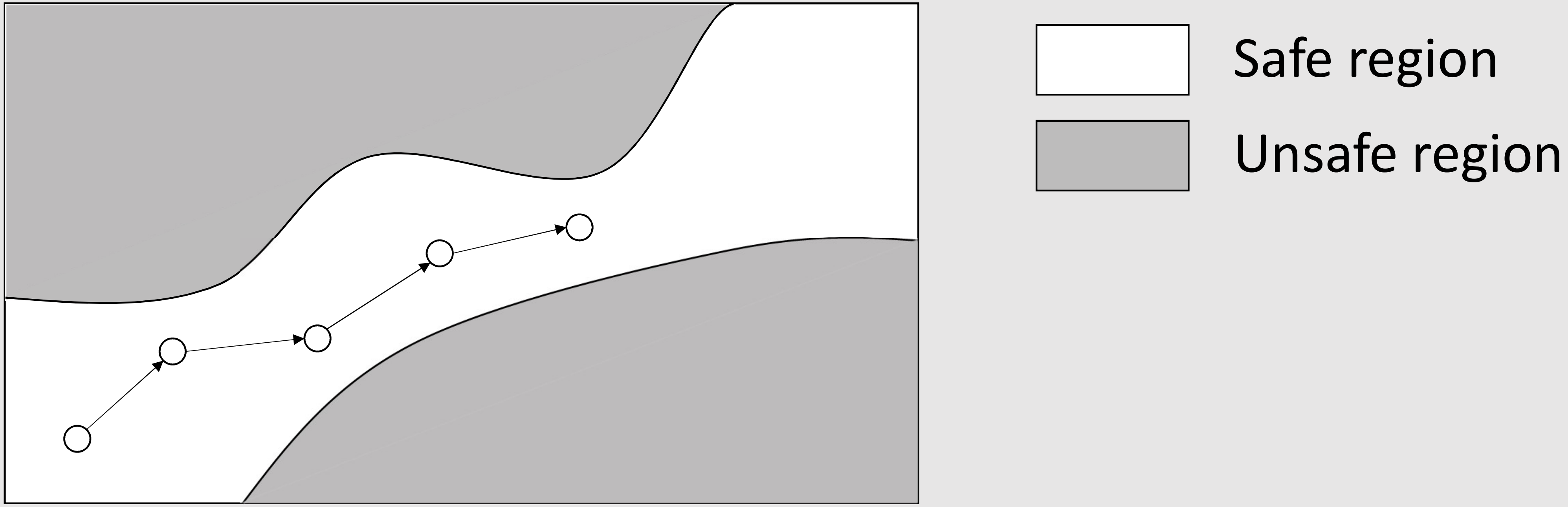
Ranking supermartingale



$\mathbb{E}\left(\begin{array}{c}\text{the value of } f\\ \text{at the next state}\end{array}\right)$ decreases at least 1

The system eventually visits ◎ almost surely

Barrier certificate



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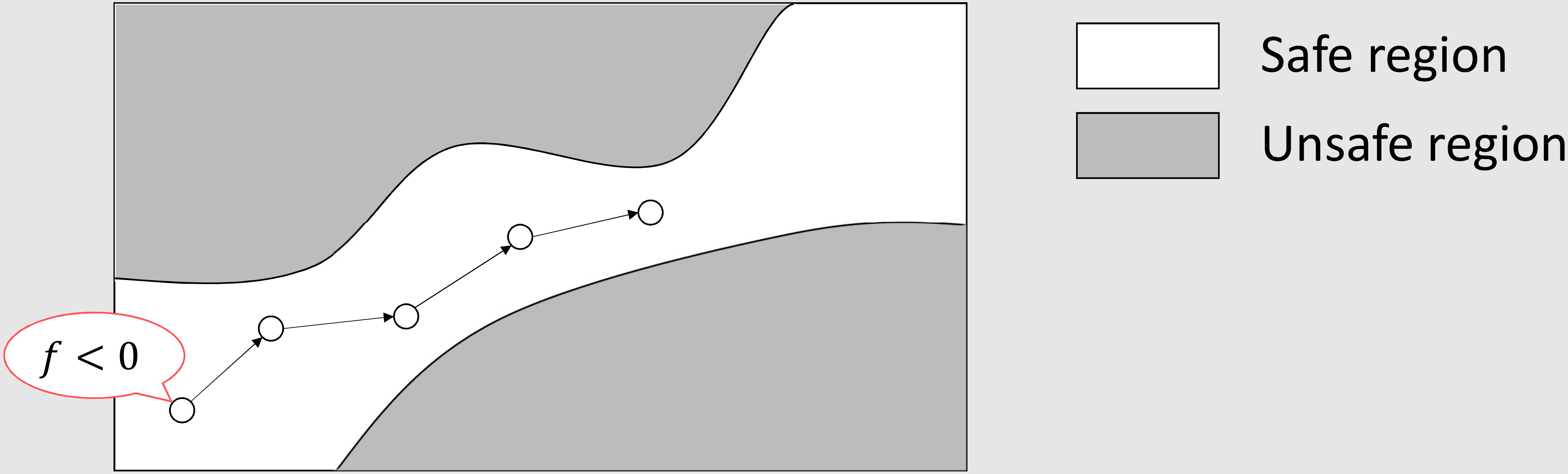

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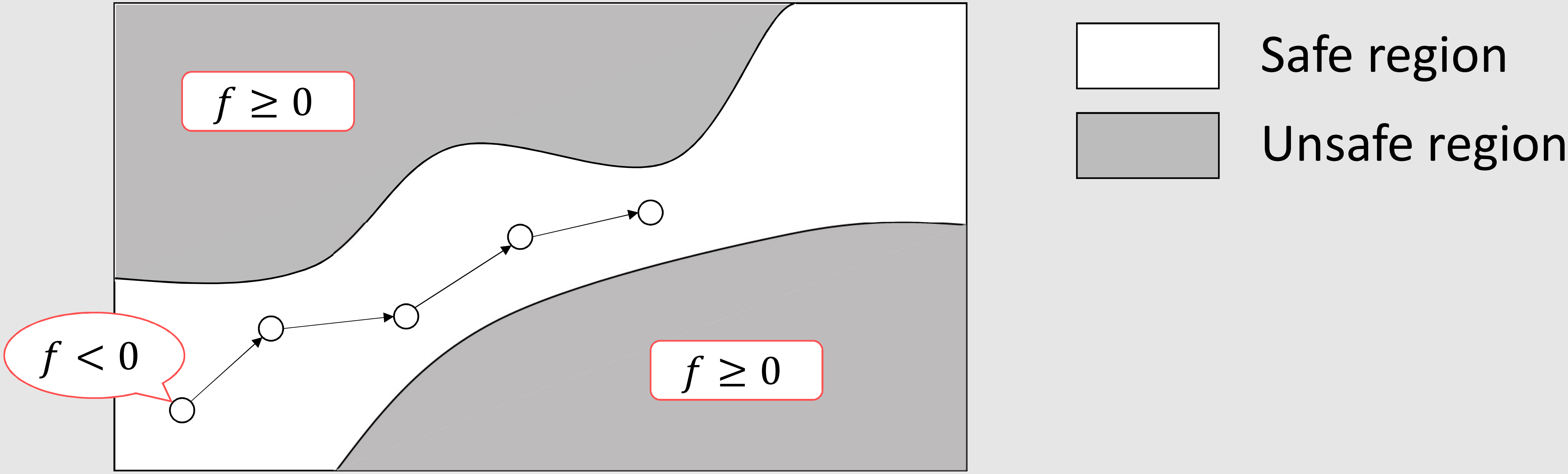

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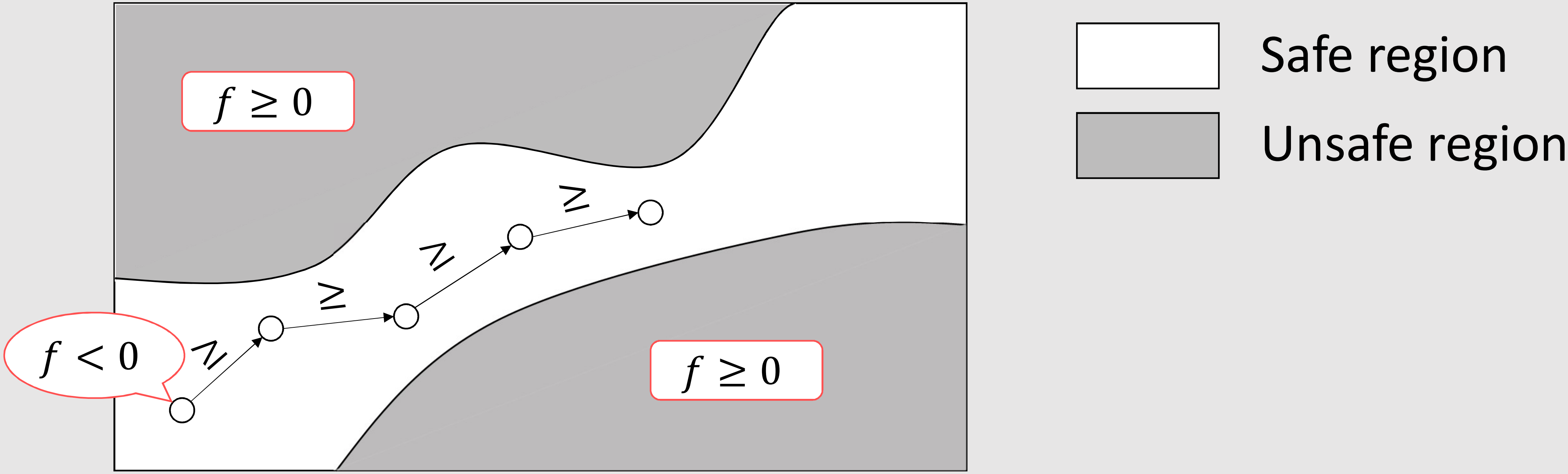

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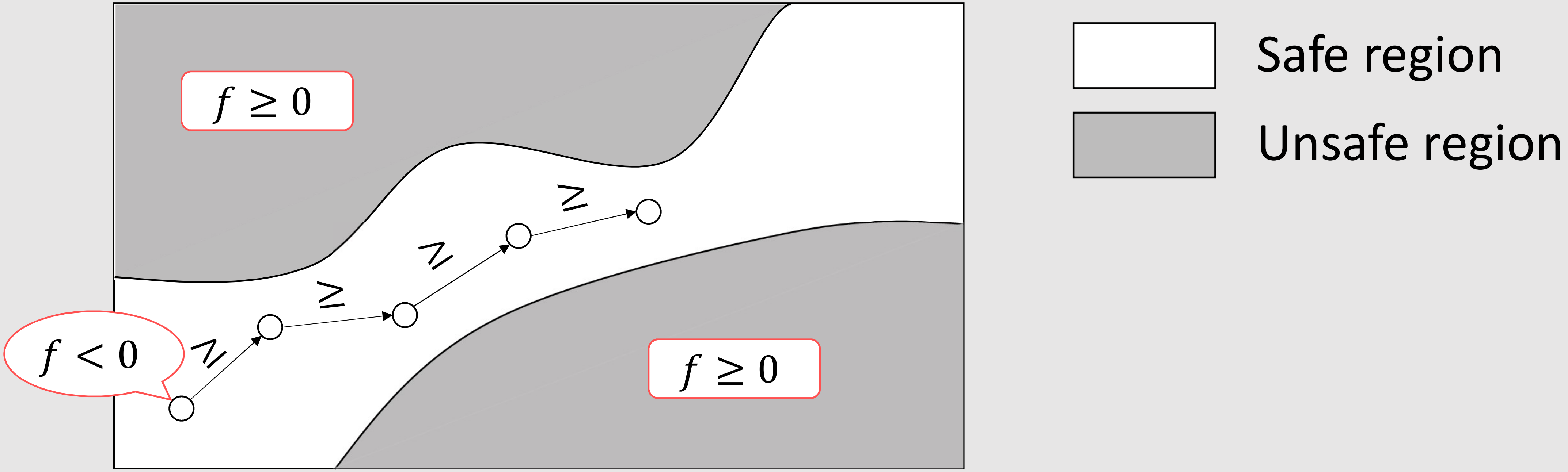
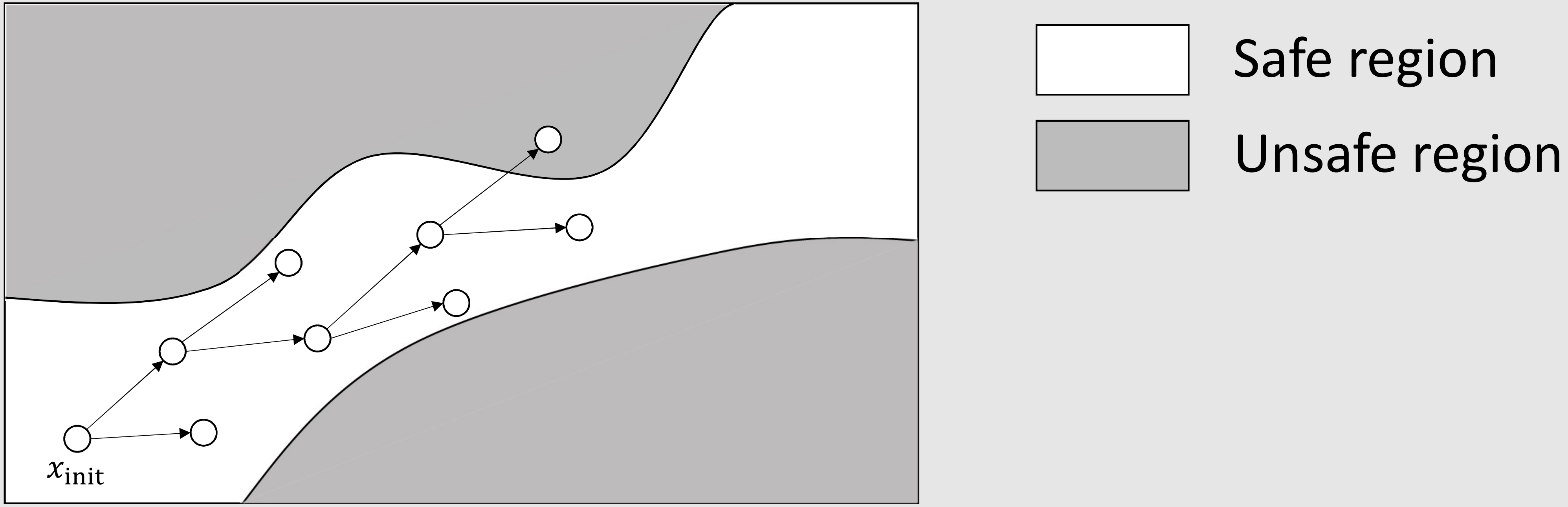


The system does not enter the unsafe region

Probabilistic barrier certificate
(a.k.a. nonnegative repulsing supermartingale)



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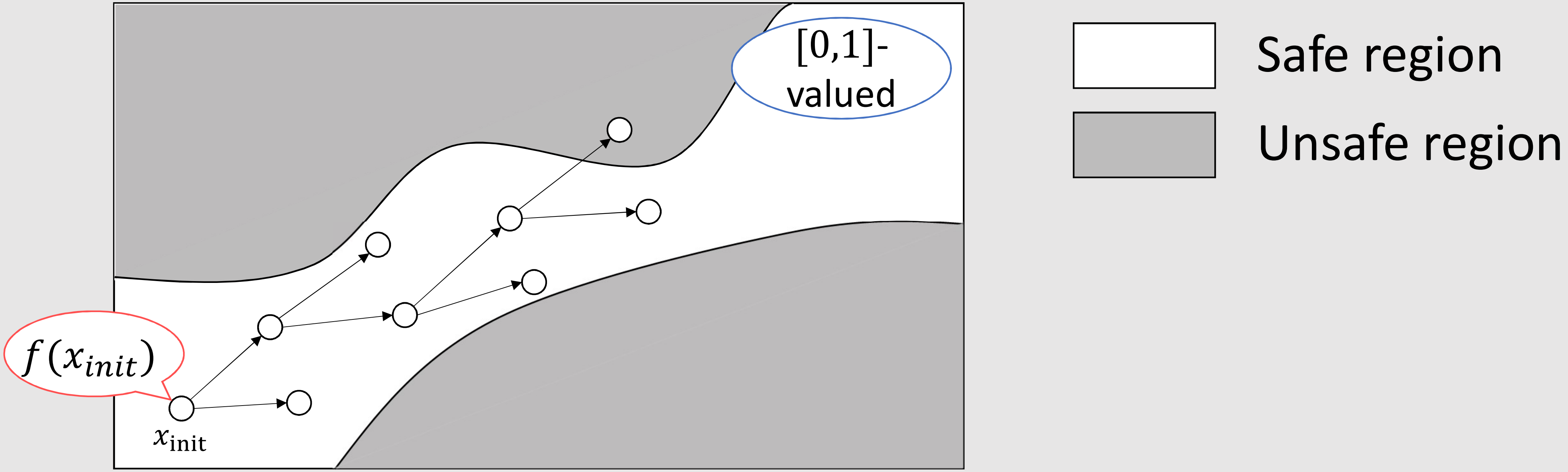

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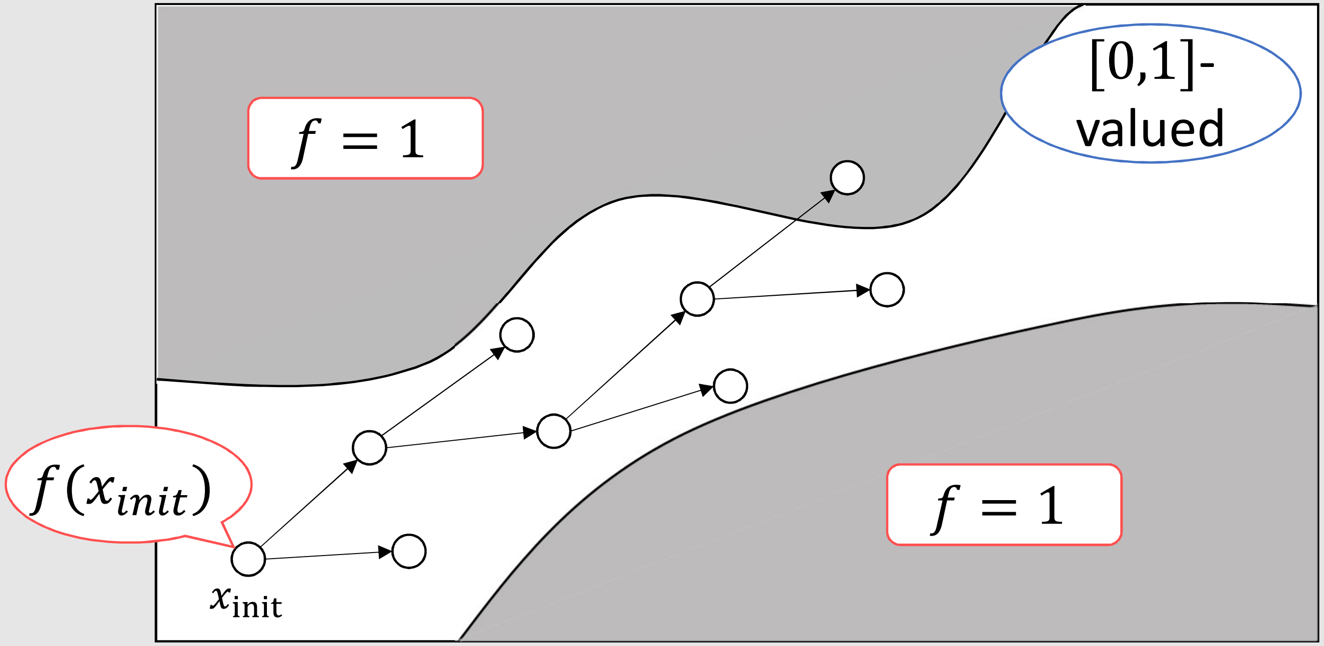

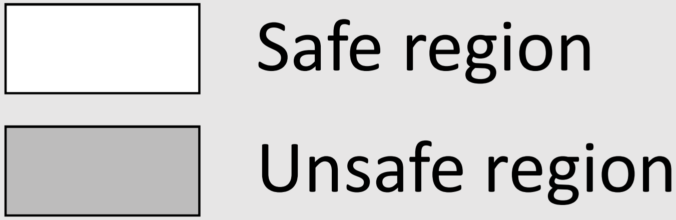

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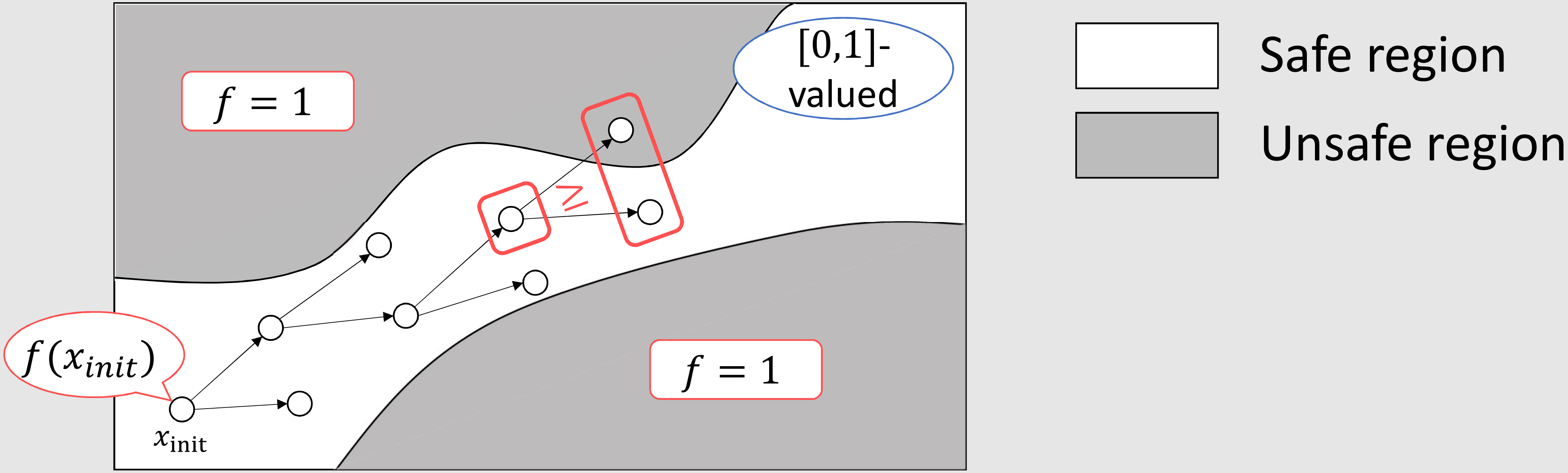

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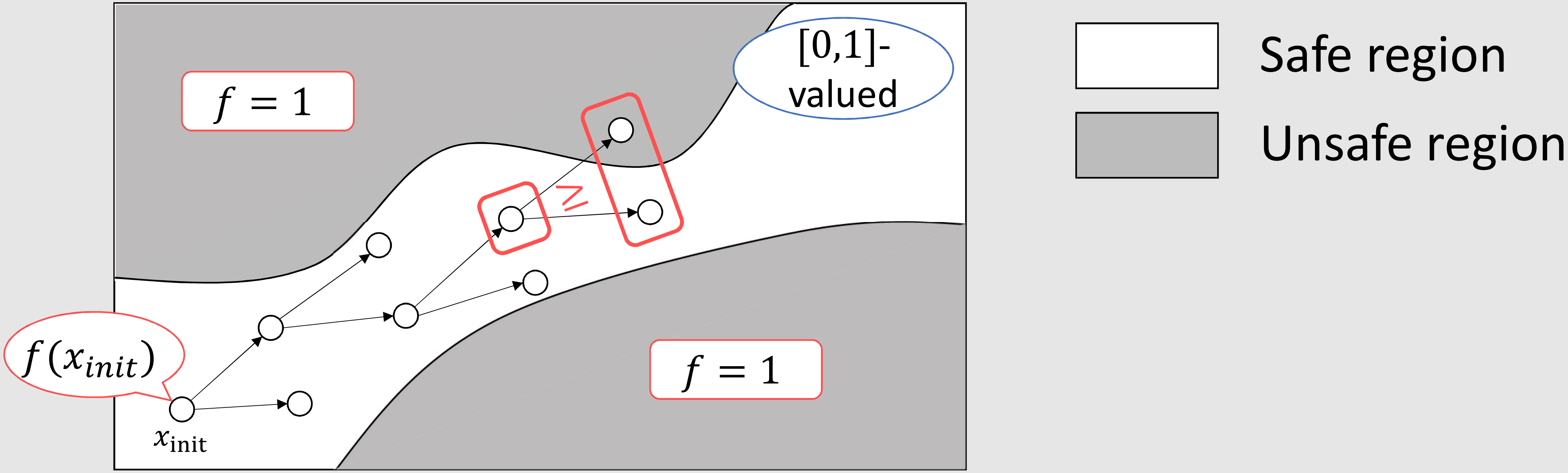

Pr(the system enters the unsafe region) $\leq f(x_{\mathrm{init}})$

Our contributions

Comprehensive account of martingale-based approximation methods via **fixed point argument**

Soundness/completeness for **uncountable-states MDPs**, under angelic/demonic nondeterminism

Implementation and **experiments**

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Implementation and experiments

Two objective functions

- Given: a control flow graph, and a subset C of its states
- $\mathbb{E}^{\text{steps}}: L \times \mathbb{R}^V \to [0, \infty]$ and $\mathbb{P}^{\text{reach}}: L \times \mathbb{R}^V \to [0,1]$ are

$$\mathbb{E}^{\text{steps}}: c \mapsto \mathbb{E}\left(\begin{matrix}\text{the number of steps from } c\\ \text{to the region } C\end{matrix}\right)$$

$$\mathbb{P}^{\text{reach}}: c \mapsto \mathbb{P}\left(\begin{matrix}\text{the system eventually visits}\\ \text{the region } C \text{ from } c\end{matrix}\right)$$

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…under angelic/demonic scheduler

Soundness/completeness

Ranking supermartingale

Soundness: $\exists(\mathrm{RankSM}) \quad \Rightarrow \quad \mathbb{E}^{\mathrm{steps}}(c_{init}) < \infty$

$(\Rightarrow \mathbb{P}^{\mathrm{reach}}(c_{init}) = 1)$

Completeness: $\mathbb{E}^{\mathrm{steps}}(c_{init}) < \infty \quad \Rightarrow \quad \exists(\mathrm{RankSM})$

Nonnegative repulsing supermartingale

Soundness: $\exists(\mathrm{RepSM}) \quad \Rightarrow \quad \mathbb{P}^{\mathrm{reach}}(c_{init}) \leq \delta$

Completeness: $\mathbb{P}^{\mathrm{reach}}(c_{init}) \leq \delta \quad \Rightarrow \quad \exists(\mathrm{RepSM})$

Soundness/completeness

Ranking supermartingale

Known — Soundness: $\exists(\mathrm{RankSM}) \Rightarrow \mathbb{E}^{\mathrm{steps}}(c_{init}) < \infty$

$(\Rightarrow \mathbb{P}^{\mathrm{reach}}(c_{init}) = 1)$

Partly known — Completeness: $\mathbb{E}^{\mathrm{steps}}(c_{init}) < \infty \Rightarrow \exists(\mathrm{RankSM})$

Nonnegative repulsing supermartingale

Partly known — Soundness: $\exists(\mathrm{RepSM}) \Rightarrow \mathbb{P}^{\mathrm{reach}}(c_{init}) \leq \delta$

Not known — Completeness: $\mathbb{P}^{\mathrm{reach}}(c_{init}) \leq \delta \Rightarrow \exists(\mathrm{RepSM})$

Soundness/completeness

For certain endofunctions Φ and Ψ,

$\mathbb{E}^{\mathrm{steps}} = \mu\Phi$ and $\mathbb{P}^{\mathrm{reach}} = \mu\Psi$

Soundness/completeness

Our theorem

$$\mathbb{E}^{\mathrm{steps}} = \mu\Phi$$

The lattice $(\mathcal{F}, \sqsubseteq)$

$\mathcal{F}$ ... the set of all (measurable) functions

$$f : L \times \mathbb{R}^V \to [0, \infty]$$

$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s. f(s) \leq g(s)$

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Soundness

$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\mathrm{steps}} \sqsubseteq f}$$

Soundness/completeness

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$$f: L \times \mathbb{R}^V \to [0, \infty]$$

$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s. f(s) \le g(s)$

Soundness

$$\frac{f \text{ is a RankSM}}{\mathbb{E}^{\mathrm{steps}} \sqsubseteq f} \quad \Leftrightarrow \quad \frac{\Phi f \sqsubseteq f}{\mu\Phi \sqsubseteq f}$$

Soundness/completeness

Our theorem

$$\mathbb{E}^{\text{steps}} = \mu\Phi$$

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Knaster-Tarski theorem

Completeness

$$\Phi\mathbb{E}^{\text{steps}} \sqsubseteq \mathbb{E}^{\text{steps}}$$

Soundness/completeness

Our theorem

$$\mathbb{P}^{\mathrm{reach}} = \mu\Psi$$

The lattice $(\mathcal{F}, \sqsubseteq)$

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$\sqsubseteq$ ... $\quad f \sqsubseteq g \Leftrightarrow \forall s. f(s) \leq g(s)$

Soundness

$$\frac{f \text{ is a RepSM}}{\mathbb{P}^{\mathrm{reach}} \sqsubseteq f} \quad \begin{matrix}\Leftrightarrow \\ \Leftrightarrow\end{matrix} \quad \frac{\Psi f \sqsubseteq f}{\mu\Psi \sqsubseteq f}$$

Knaster-Tarski theorem

Completeness

$$\Psi\mathbb{P}^{\mathrm{reach}} \sqsubseteq \mathbb{P}^{\mathrm{reach}}$$

Our contributions

Comprehensive account of martingale-based approximation methods via fixed point argument

Soundness/completeness for uncountable-states MDPs, under angelic/demonic nondeterminism

Implementation and experiments

Soundness/completeness for martingale methods

Approximation method	It certifies	Soundness	Completeness
Additive ranking Supermartingale (Chakarov-Sankaranarayanan, CAV'13 etc.)	$\mathbb{E}^{\mathrm{steps}} < \infty$ $(\mathbb{P}^{\mathrm{reach}} = 1)$	Yes (MDP, continuous variable)	Yes (MDP, discrete variable)
Nonnegative repulsing supermartingale (Steinhardt+, IJRR'12 etc.)	$\mathbb{P}^{\mathrm{reach}} \leq \delta$	Yes (Markov Chain)	-
γ-scaled submartingale (Urabe+, LICS'17)	$\mathbb{P}^{\mathrm{reach}} \geq \delta$	Yes (Markov Chain)	-
ε-decreasing repulsing supermartingale (Chatterjee+, POPL'17)	$\mathbb{P}^{\mathrm{reach}} \leq \delta$	Yes (MDP, continuous variable, linearity assumpt.)	-

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Implementation and experiments

Implementation and experiments

		Prog. I (linear)		Prog. II (deg.-2 poly.)		Prog. II (deg.-3 poly.)	
	param.	time (s)	bound	time (s)	bound	time (s)	bound
(a-1)	$p_1 = 0.2$ $p_2 = 0.4$	0.021	≤ 0.825	530.298	≤ 0.6552	572.393	≤ 0.6555
	$p_1 = 0.8$ $p_2 = 0.1$	0.024	≤ 1	526.519	≤ 1.0	561.327	≤ 1.0

Table 1. Bounds by U-NNRepSupM

①

	true reachability probability	U-NNRepSupM	1-RepSupM
(c-1)	$\frac{(0.4/0.6)^5-(0.4/0.6)^{10}}{1-(0.4/0.6)^{10}} \approx 0.116$	0.505	< 1
(c-2)	0.5	0.5	—
(c-3)	$\int_0^1 (\frac{0.25}{0.75})^{\lceil \log_2(1/x) \rceil} dx \approx 0.2$	0.5	—
(c-4)	$(\frac{0.25}{0.75})^1 \approx 0.333$	—	< 1

②

Table 3. Probabilistic bounds given by U-NNRepSupM and ε-RepSupM

		Prog. III (linear)	
	param.	time (s)	bound
(a-1)	$p_1 = 0.2$ $p_2 = 0.4$	0.026	≥ 0
	$p_1 = 0.8$ $p_2 = 0.1$	0.022	≥ 0.751
(a-2)	$M_1 = -1$ $M_2 = 2$	0.033	≥ 0
	$M_1 = -2$ $M_2 = 1$	0.033	≥ 0.767
(a-3)	$M_1 = -1$ $M_2 = 2$	0.028	≥ 0
	$M_1 = -2$ $M_2 = 1$	0.040	≥ 0.801
(b)	$c = 0.1$ $p = 0.5$	0.056	≥ 0
	$c = 0.1$ $p = 0.1$	0.054	≥ 0.148

①

Table 2. Bounds by L-γ-SclSubM with $\gamma = 0.999$

- Implemented template-based synthesis algorithms
- Nontrivial bounds are found (①)
- Observed comparative advantage of nonnegative RepSM over ε-decreasing RepSM (②)

Summary

- Martingale can evaluate reachability of probabilistic programs in various ways
- We gave a comprehensive account of martingale-based approximation methods via fixed point argument
- We proved soundness/completeness of several methods for uncountable-states MDPs, which extends known results
- We demonstrated implementation and experiments